

RANDOM WALKS, KLEINIAN GROUPS, AND BIFURCATION CURRENTS

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ABSTRACT. Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of a finitely generated group G into $\mathrm{PSL}(2, \mathbb{C})$, parameterized by a complex manifold Λ . We define a notion of *bifurcation current* in this context, that is, a positive closed current on Λ describing the bifurcations of this family of representations in a quantitative sense. It is the analogue of the bifurcation current introduced by DeMarco for holomorphic families of rational mappings on \mathbb{P}^1 . Our definition relies on the theory of random products of matrices, so it depends on the choice of a probability measure μ on G .

We show that under natural assumptions on μ , the support of the bifurcation current coincides with the bifurcation locus of the family. We also prove that the bifurcation current describes the asymptotic distribution of several codimension 1 phenomena in parameter space, like accidental parabolics or new relations, or accidental collisions between fixed points.

1. INTRODUCTION

In recent years, the use of techniques from higher dimensional holomorphic dynamics, especially positive currents, has led to interesting new insights on the structure of parameter spaces of holomorphic dynamical systems on the Riemann sphere. There is a deep and fruitful analogy –first brought to light by Sullivan [Su1]– between holomorphic dynamics on \mathbb{P}^1 and the theory of Kleinian groups. Our purpose in this paper is to develop these ideas on the Kleinian group side, by initiating the study of *bifurcation currents* in this setting.

Let us first briefly discuss this notion in the context of rational mappings. Let Λ be a complex manifold and $f = (f_\lambda)_{\lambda \in \Lambda} : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be a holomorphic family of rational maps of fixed degree. The simplest way to define a positive closed current on Λ associated to this family is, following DeMarco [DeM1, DeM2], to observe that the Lyapunov exponent $\chi(f_\lambda)$ of f_λ relative to its unique measure of maximal entropy defines a plurisubharmonic (psh for short) function on Λ . We then put $T_{\mathrm{bif}} = dd^c(\chi(f_\lambda))$. This is by definition the *bifurcation current* of the family. In the most studied family $(z^2 + \lambda)_{\lambda \in \mathbb{C}}$ of quadratic polynomials, T_{bif} is just the harmonic measure of the Mandelbrot set.

DeMarco proved that $\mathrm{Supp}(T_{\mathrm{bif}})$ is exactly the bifurcation locus Bif (defined, e.g., as the locus of parameters where the Julia set does not move continuously in the Hausdorff topology). A word about the proof: the inclusion $\mathrm{Supp}(T_{\mathrm{bif}}) \subset \mathrm{Bif}$ is obvious, while the converse inclusion is based on a formula for the Lyapunov exponent in terms of the dynamics of critical points. It follows in particular that if the Lyapunov exponent is pluriharmonic in some region of parameter space, then the critical points cannot bifurcate there. Standard arguments then imply that the dynamics is stable.

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A remarkable feature of the bifurcation current is that it describes the asymptotic distribution of natural sequences of dynamically defined subvarieties of parameter space. For instance it was shown by Favre and the second author in [DF] that the hypersurfaces of parameters possessing a preperiodic critical point (of preperiod n tending to infinity) equidistribute towards T_{bif} . Another result, due to Bassanelli and Berteloot, asserts that parameters admitting a periodic point of period n and a given multiplier typically equidistribute towards T_{bif} [BB2, BB3].

Let us now turn to the subject of the paper. Let Λ be a (connected) complex manifold, G be a finitely generated group and $\rho = (\rho_\lambda)_{\lambda \in \Lambda} : \Lambda \times G \rightarrow \text{PSL}(2, \mathbb{C})$ be a holomorphic family of non-elementary representations of G . To avoid trivialities, assume further that the representations ρ_λ are faithful at generic parameters, and not all conjugate to each other (in $\text{PSL}(2, \mathbb{C})$). For such a family of Möbius subgroups, there is a well-established notion of bifurcation, mainly due to Sullivan [Su2] (see also [Ber]), defined (by the negative) by saying that an open set U is contained in the stability locus¹ if for every $g \in G$, the holomorphic family of Möbius transformations $\rho_\lambda(g)$ is of constant type (loxodromic, parabolic or elliptic) throughout U . In particular the fixed points $\rho_\lambda(g)$ can be followed holomorphically over U . Using the theory of holomorphic motions, Sullivan proved that representations in a stable family are quasi-conformally conjugate on \mathbb{P}^1 . He also proved that these representations must then be discrete with a non-empty set of discontinuity.

To associate a bifurcation current to such a family we use the theory of random walks on groups. For this, we choose a probability measure μ on G (satisfying certain natural technical assumptions that will be made clear in the text), and consider the random walk on G whose transition probabilities are given by μ . For the sake of simplicity we may suppose in this introduction that μ is equidistributed on a finite symmetric set of generators of G , in which case we are just considering the simple random walk on the associated Cayley graph.

Given a representation ρ of G into $\text{PSL}(2, \mathbb{C})$ we can now define a Lyapunov exponent by the formula

$$\chi(\rho) := \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|\rho(g)\| d\mu^n(g).$$

Here $\|\cdot\|$ refers to any matrix norm on $\text{PSL}(2, \mathbb{C})$, and μ^n denotes the n^{th} convolution power of μ , that is, the image of the product measure $\mu^{\otimes n}$ on G^n under the map $(g_1, \dots, g_n) \mapsto g_1 \cdots g_n$.

For a holomorphic family of representations as above, we obtain in this way a non-negative psh function $\lambda \mapsto \chi(\rho_\lambda)$, and define, in analogy with the polynomial case, the bifurcation current by the formula $T_{\text{bif}} = dd^c(\chi(\rho_\lambda))$.

For readers not necessarily familiar with positive currents, it is worth noting that our results are already interesting when $\dim(\Lambda) = 1$, in which case one can simply replace “psh” by “subharmonic” and “positive current” by “positive measure”. Nevertheless, some arguments in the proof require to work with actual currents on the 2-dimensional space $\Lambda \times \mathbb{P}^1$.

The theory of random products of matrices will be used to study the properties of this Lyapunov exponent function, and show that the bifurcation current is a meaningful object, truly capturing the bifurcations of the family.

A first fundamental result, due to Furstenberg, asserts that under the above assumptions, $\chi(\rho_\lambda)$ is positive, and depends continuously on λ . Furthermore, $\chi(\rho_\lambda)$ admits an expression

¹One should be careful not to be confused with the notion of a stable representation in the sense of geometric invariant theory. In this paper, stability will always be understood in the sense of dynamical systems.

in terms of a canonical probability measure ν_λ on \mathbb{P}^1 , invariant under the average action of $\rho_\lambda(G)$, which will play an important role in the paper. More generally, it is remarkable that the proofs in the paper will require non-trivial results like the exponential convergence of the transition operator, the Large Deviations Theorem, etc. (see Bougerol-Lacroix [BL] and Furman [Furm] for good introductory texts on these topics)

Our first main result is the characterization of the support of the bifurcation current.

Theorem A. *Let (G, ρ, μ) be a holomorphic family of representations as above. Then the support of T_{bif} is equal to the bifurcation locus.*

To say it differently, the stability of a holomorphic family of Möbius subgroups is equivalent to the pluriharmonicity of the Lyapunov exponent function (for any μ). A notable consequence of the theorem is that the support of T_{bif} does not depend on μ . Another corollary, which was a basic source of motivation in [DeM1], is that if Λ is a Stein manifold, the components of the stability locus are also Stein. This holds in particular when Λ is the space of all representations of G into $\text{PSL}(2, \mathbb{C})$ modulo conjugacy, which is an affine algebraic variety². As a corollary one recovers the result of Bers and Ehrenpreis [Ber-E] that Teichmüller spaces are Stein manifolds.

As before, the delicate inclusion in Theorem A is to show that if χ is pluriharmonic on an open subset U , then U must be contained in the stability locus. The approach used in the context of rational dynamics seems to have no analogue here. Instead, we use a geometric interpretation of the bifurcation current, which we briefly describe now (the details can be found in §3.2).

Let us look at the fibered action of G on $\Lambda \times \mathbb{P}^1$: for this, we fix $z_0 \in \mathbb{P}^1$ and consider the graphs over Λ defined by $\lambda \mapsto g_\lambda(z_0)$ as g ranges in G . Over the stability locus, these graphs form a normal family. On the other hand, they tend to oscillate wildly when approaching the bifurcation locus. We show that T_{bif} describes the asymptotic distribution of this oscillation phenomenon (see Theorem 3.5 for a precise statement). In particular, $\lambda_0 \in \text{Supp}(T_{\text{bif}})$ if and only if for every neighborhood $U \ni \lambda_0$, the average volume, with respect to μ^n , of the graph of $\lambda \mapsto g_\lambda(z_0)$ over U , grows linearly with n (which is the fastest possible growth).

On the other hand we show that if U is disjoint from $\text{Supp}(T_{\text{bif}})$, the average volume of these graphs is locally bounded. Using Bishop's compactness theorem for sequences of analytic sets, this allows us to construct for every $\lambda \in U$ an equivariant map θ_λ from the Poisson boundary of (G, μ) to \mathbb{P}^1 , depending holomorphically on λ , which ultimately turns out to be a holomorphic motion. Sullivan's theory then implies that the family of representations is stable over U .

It is an easy consequence of Sullivan's results that for every $t \in [0, 4]$, the set of parameters λ_0 such that there exists $g \in G$ with non-constant trace and $\text{tr}^2(g_{\lambda_0}) = t$ is dense in the bifurcation locus (see Corollary 2.7 below). The same is true for the set of parameters at which a collision between fixed points of different elements occurs. In the light of what is known about spaces of rational maps, it is natural to wonder whether in these assertions, density can be turned into equidistribution. This will be the second main theme developed in the paper.

If V is an analytic subset of Λ , recall that the integration current on V is denoted by $[V]$. When $\dim(\Lambda) = 1$ (hence $\dim(V) = 0$) this is just a sum of Dirac masses at the points of V (counted with multiplicities, if any). It is convenient to adopt the convention that $[\Lambda] = 0$.

²Notice that no trouble can arise from the possible singularities of these varieties, since the components of the stability locus are disjoint from them (see [Kap, §8.8]).

Our first equidistribution theorem concerns random sequences in G .

Theorem B. *Let (G, ρ, μ) be a holomorphic family of representations as above. Consider the product space $G^{\mathbb{N}}$, endowed with the product measure $\mu^{\mathbb{N}}$. Then the following conclusions hold.*

- (1) *For $g \in G$ and $t \in \mathbb{C}$, let $Z(g, t) = \{\lambda, \text{tr}^2(g_\lambda) = t\}$. Then for $\mu^{\mathbb{N}}$ -a.e. sequence $(g_n)_{n \geq 1}$ we have that*

$$\frac{1}{2n} [Z(g_n \cdots g_1, t)] \xrightarrow[n \rightarrow \infty]{} T_{\text{bif}}.$$

- (2) *For $g, h \in G$, let $F(g, h)$ be the analytic subset of Λ defined by the condition that g_λ and h_λ have a common fixed point. Then for $\mu^{\mathbb{N}} \otimes \mu^{\mathbb{N}}$ -a.e. pair $((g_n), (h_n))$,*

$$\frac{1}{4n} [F(g_n \cdots g_1, h_n \cdots h_1)] \xrightarrow[n \rightarrow \infty]{} T_{\text{bif}}.$$

It follows in particular from (1) that if the bifurcation locus is non empty, then almost surely $Z(g_n \cdots g_1, t)$ is a non empty proper analytic subvariety for large n (and similarly for (2)).

As far as we know, this is the first equidistribution statement of this kind. The proof is based on a general machinery which produces equidistribution theorems in parameter space from limit theorems for random sequences at every (fixed) parameter (see Theorem 4.1).

Since the 1980's, several authors have produced pictures of stability loci in 1-dimensional families of representations, by plotting solutions of $\text{tr}^2(g) = 4$ in parameter space (see [MSW, Chapter 10] for a beautiful account on this). These pictures exhibit intriguing accumulation patterns as the length of g increases. Our equidistribution results say that these patterns are governed by the bifurcation measure –at least when the words $g \in G$ are chosen according to a random walk on G .

Here is a consequence of Theorem B which does not make explicit reference to a measure on G , and does not seem easy to prove without using probabilistic methods: for any $\varepsilon > 0$ and any relatively compact set $\Lambda' \subset \Lambda$, there exists $g \in G$ such that the set of parameters λ such that $\rho_\lambda(g) = \text{id}$ (resp. $\text{tr}^2(\rho_\lambda(g)) = 4$) is ε -dense in the bifurcation locus restricted to Λ' . For this, it suffices to take $t = 0$ (resp. $t = 4$) in the first item of the above theorem, and take $g = (g_n \cdots g_1)^4$ (resp. $g = g_n \cdots g_1$) for a $\mu^{\mathbb{N}}$ -generic sequence (g_n) and large enough n .

We are also able to estimate the speed of convergence in item (1) above after some averaging with respect to g . This requires some additional assumptions on Λ .

Theorem C. *Let (G, ρ, μ) be a holomorphic family of representations as above, and fix $t \in \mathbb{C}$. Suppose in addition that one of the following conditions holds.*

- i. *either Λ is an algebraic family of representations, defined over $\overline{\mathbb{Q}}$.*
- ii. *or there is at least one geometrically finite representation in Λ .*

Then there exists a constant C such that for every test form ϕ

$$\left\langle \frac{1}{2n} \int [Z(g, t)] d\mu^n(g) - T_{\text{bif}}, \phi \right\rangle \leq C \frac{\log n}{n} \|\phi\|_{C^2}.$$

The proof is more involved than that of Theorem B, and based on several interesting ingredients. To prove the theorem, we need to understand how the potential of $\frac{1}{2n} \int [Z(g, t)]$, namely $\frac{1}{2n} \int \log |\text{tr}^2(g_\lambda) - t| d\mu^n(g)$ is close (in $L^1_{\text{loc}}(\Lambda)$) to the Lyapunov exponent function χ . Two main ingredients for this are:

- precise (large deviations) estimates on the asymptotic distribution of $\text{tr}^2(g_\lambda)$ for fixed λ (which are established in Appendix A),
- bounds on the volume of the set of representations possessing elements with trace close to t .

The role of the additional assumptions *i.* and *ii.* is, for the purpose of establishing these volume bounds, to ensure that $\log |\text{tr}^2(g_\lambda) - t|$ cannot be uniformly close to $-\infty$ along Λ . For instance, under *ii.* we have a good control of the set of values of $\text{tr}^2(g_\lambda)$ at the geometrically finite parameter. The result then follows from classical estimates on the volume of sub-level sets of psh functions. We also see that we can weaken assumption *ii.* by only requiring that the family of representations (ρ_λ) can be analytically continued to a family containing a geometrically finite representation.

Under *i.*, the desired estimate on $\log |\text{tr}^2(g_\lambda) - t|$ follows from number-theoretic considerations.

As a byproduct of our methods, we obtain a new proof and a generalization of a result of Kaloshin and Rodnianski [KR] (see Remark 4.18).

It is unclear whether the speed $O\left(\frac{\log n}{n}\right)$ that we obtain is optimal or not. One might guess that the optimal speed is bounded below by $O\left(\frac{1}{n}\right)$ (see Remark 3.12).

For the analogous equidistribution theorems associated to families of rational maps, no such general estimate is known. The only quantitative equidistribution result towards T_{bif} that we know of in that context is specific to the unicritical family $z^d + c$, and relies on number-theoretic ideas [FRL, Theorem 5]. Notice also that the proofs of most of the equidistribution theorems in [DF, BB3] also require some global assumptions on Λ . In whatever case, it is unclear whether these assumptions are really necessary.

Many particular families of representations have been studied in the literature. Let us focus on one classical situation (more details can be found in §5.1 which is itself a preview of a sequel [DD] to this paper). Fix a compact Riemann surface S of genus $g \geq 2$, and introduce the complex affine space $\Lambda \simeq \mathbb{C}^{3g-3}$ of complex projective structures on S , compatible with the complex structure of S . Any such projective structure gives rise to a *monodromy representation* of the fundamental group of S into $\text{PSL}(2, \mathbb{C})$, that varies holomorphically with λ . The well-known *Bers slices* of Teichmüller space are obtained from this construction.

Any random walk on the group $\pi_1(S)$ then gives rise to a bifurcation current on Λ . In fact in this setting there is more: we claim that there exists a *canonical bifurcation current* on Λ . For this, let us shift a little bit our point of view, and instead of a discrete random walk on $\pi_1(S)$, consider the Brownian motion on S (which depends only on the Riemann surface structure). A projective structure being given, we can consider the growth rate of its holonomy over generic Brownian paths, thereby obtaining a Lyapunov exponent, in the spirit of [DK]. This induces a natural psh function on Λ , hence a natural bifurcation current. It turns out that this bifurcation current is induced by a measure on $\pi_1(S)$, therefore it satisfies the above theorems.

There is some similarity between Theorem A and a recent result of Avila's [Av], appearing as a crucial step in the proof of the stratified analyticity of the Lyapunov exponent of quasi-periodic Schrödinger operators. To be precise, to an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and a real-analytic function $A : \mathbb{R}/\mathbb{Z} \rightarrow \text{PSL}(2, \mathbb{C})$, we associate a Lyapunov exponent by the

formula

$$L(A, \alpha) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \log \|A(x)A(x+\alpha) \cdots A(x+(n-1)\alpha)\| dx.$$

The variation of the function $\varepsilon \mapsto L(A_\varepsilon, \alpha)$, where $A_\varepsilon(\cdot) = A(\cdot + i\varepsilon)$, is studied in detail, and Avila proves [Av, Theorem 6] that if $L(\alpha, A) > 0$, this function is locally affine if and only if (α, A) is uniformly hyperbolic. This is completely analogous to the above statement that a family of representations is stable if and only if the Lyapunov exponent is pluriharmonic.

It is also worth mentioning the recent work of Cantat [Ca] in which the author uses higher dimensional holomorphic dynamics to study the action of the mapping class group on the character variety of the once-punctured torus (resp. the four times punctured sphere). A given mapping class acts by holomorphic automorphisms on the character variety, so it usually admits invariant currents, supported on the bifurcation locus. It is unclear to us whether these currents are related to ours. It would be interesting nevertheless to explore the relationship between the two constructions.

Here is the structure of the paper. In Section 2 we give some background on holomorphic families of subgroups of $\mathrm{PSL}(2, \mathbb{C})$, as well as a number of basic results in the theory of random products of matrices. In Section 3 we introduce the bifurcation current, give its geometric interpretation and prove Theorem A. We also give in Theorem 3.14 the classification of all “stationary currents” for a holomorphic family of Möbius groups (under a mild assumption). Section 4 is mainly devoted to Theorems B and C. Two auxiliary results required in the proof of the equidistribution theorems have been moved to appendices: in Appendix A, we study the distribution of fixed points of random products in $\mathrm{PSL}(2, \mathbb{C})$ (more general related results recently appeared in [Ao]). In appendix B we prove a number-theoretic lemma related to the assumption *i.* of Theorem C. Finally, in Section 5 we outline some further developments (in particular the construction of canonical bifurcation currents), as well as a number of open questions.

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2. PRELIMINARIES

2.1. Möbius subgroups. Mainly for the purpose of fixing notation, we recall some basics on subgroups of $\mathrm{Aut}(\mathbb{P}^1)$, where \mathbb{P}^1 will refer to the Riemann sphere. The reader is referred to e.g. [Bea, Kap] for more details.

We identify $\mathrm{Aut}(\mathbb{P}^1) = \left\{ z \mapsto \frac{az+b}{cz+d}, ad-bc \neq 0 \right\}$ with the matrix group $\mathrm{PSL}(2, \mathbb{C})$. If $\gamma \in \mathrm{PSL}(2, \mathbb{C})$, it is often convenient for calculations to lift γ to one of its representatives in $\mathrm{SL}(2, \mathbb{C})$. We define the quantities $\|\gamma\|$ and $\mathrm{tr}^2 \gamma$ by lifting γ to one of its representatives in $\mathrm{SL}(2, \mathbb{C})$. Of course the result does not depend on the lift. In this paper $\|\gamma\|$ denotes the operator norm associated to the Hermitian norm on \mathbb{C}^2 . As it is well-known $\|\gamma\|$ equals the square root of the spectral radius of A^*A , where A is any matrix representative of γ . It will also be sometimes convenient to work with $\|\gamma\|_2 := (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{1/2}$.

As usual we classify Möbius transformations into three types:

- *parabolic* if $\mathrm{tr}^2(\gamma) = 4$ and $\gamma \neq \mathrm{id}$; it is then conjugate to $z \mapsto z + 1$;

- *elliptic* if $\text{tr}^2(\gamma) \in [0, 4)$, it is then conjugate to $z \mapsto e^{i\theta}z$ for some real number θ , and $\text{tr}^2 \gamma = 2 + 2 \cos(\theta)$.
- *loxodromic* if $\text{tr}^2(\gamma) \notin [0, 4]$, it is then conjugate to $z \mapsto kz$, with $|k| \neq 1$.

We equip \mathbb{P}^1 with the spherical metric $\frac{|dz|}{1+|z|^2}$, and the associated spherical volume form, simply denoted by dz . The subgroup of elements of $\text{PSL}(2, \mathbb{C})$ that preserve this metric is isomorphic to $\text{SO}(3, \mathbb{R})$. As usual, these elements are called rotations.

The following elementary lemma shows that when the fixed points of a loxodromic map γ are separated enough, the quantities $\sqrt{|\text{tr}^2 \gamma|}$ and $\|\gamma\|$ are essentially the same. We use the following notation: if u and v are two real valued functions, we write $u \asymp v$ if there exists a constant $C > 0$ such that $\frac{1}{C}u \leq v \leq Cu$.

Lemma 2.1. *If γ is not parabolic, then*

$$\|\gamma\| \asymp \max(1, \frac{\sqrt{|\text{tr}^2 \gamma - 4|}}{\delta})$$

where δ is the distance between the two fixed points of γ .

Proof. Let $\gamma(z) = \frac{az+b}{cz+d}$ as before. Pick a fixed point of γ with multiplier of smallest modulus, and conjugate by a rotation so that this fixed point becomes ∞ . This does not affect the trace nor the norm of γ . The expression of γ is now $\gamma(z) = a^2z + ab$, with $|a| \geq 1$. The other fixed point of γ is $ab/(1 - a^2)$ (by assumption, $a^2 \neq 1$). It will be convenient to assume that this point is separated from 0 by a certain distance, say 1. To achieve this, we conjugate γ by a translation τ of bounded length. Of course, $\|\tau^{-1}\gamma\tau\| \asymp \|\gamma\|$, so it is enough to estimate the norm of $\tau^{-1}\gamma\tau$, which we rename as γ .

This being done, we have the following formulas

- $\text{tr}^2 \gamma = (a + \frac{1}{a})^2$,
- $\|\gamma\| \asymp \|\gamma\|_2 = \sqrt{|a|^2 + \frac{1}{|a|^2} + |b|^2}$,
- $\delta \asymp \frac{|1-a^2|}{|ab|}$.

We split the argument into two cases. First suppose that $|a|$ is large, $|a| \geq 1000$, say. Then, we have that

$$\text{tr}^2 \gamma \asymp a^2, \quad \delta \asymp \frac{|a|}{|b|}, \quad \|\gamma\| \asymp \frac{|a|}{\delta},$$

and the lemma is proved in this case because $\frac{\text{tr}^2 \gamma}{\delta^2}$ is large.

Now, suppose that $|a| \leq 1000$. As before, $\delta \asymp \frac{|1/a - a|}{|b|}$, and since $(a - 1/a)^2 = \text{tr}^2 \gamma - 4$, we get that $\|\gamma\|^2 \asymp 1 + |b|^2 \asymp 1 + \frac{\text{tr}^2 \gamma - 4}{\delta^2}$, and the lemma follows. \square

For $z_0 \in \mathbb{P}^1$ and $\gamma \in \text{PSL}(2, \mathbb{C})$, let $\|\gamma z_0\| = \frac{\|\gamma Z_0\|}{\|Z_0\|}$, where $Z_0 \in \mathbb{C}^2$ is any lift of z_0 and $\|\cdot\|$ in \mathbb{C}^2 is the Hermitian norm. We have the following estimate:

Lemma 2.2. *There exists a universal constant C such that if $\gamma \in \text{PSL}(2, \mathbb{C})$,*

$$\left| \int_{\mathbb{P}^1} \log \|\gamma z_0\| dz_0 - \log \|\gamma\| \right| \leq C.$$

Proof. Let $A \in \text{SL}(2, \mathbb{C})$ be a matrix representative of γ . By the KAK decomposition there exists $R, R' \in \text{SU}(2)$ and $B = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ such that $A = RBR'$, where $\sigma = \|A\|$ is the spectral radius of $\sqrt{A^*A}$. Changing variables, we see that $\int_{\mathbb{P}^1} \log \|\gamma z_0\| dz_0 = \int_{\mathbb{P}^1} \log \|Bw_0\| dw_0$.

Therefore it is enough to prove the lemma with B in place of γ , which will be left as an exercise to the reader. \square

The following fact will also be useful (notice that if $\gamma, \gamma' \in \mathrm{PSL}(2, \mathbb{C})$, the trace $\mathrm{tr}[\gamma, \gamma']$ is well-defined).

Lemma 2.3 ([Bea, Thm. 4.3.5]). *Two Möbius transformations γ and γ' have a common fixed point in \mathbb{P}^1 if and only if $\mathrm{tr}[\gamma, \gamma'] = 2$.*

Recall that the action of a Möbius transformation on \mathbb{P}^1 naturally extends to the 3-dimensional hyperbolic space \mathbb{H}^3 . Let now Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. We say that Γ is *elementary* if it admits a finite orbit in $\overline{\mathbb{H}^3}$. Then, either Γ fixes a point in \mathbb{H}^3 and is conjugate to a subgroup of $\mathrm{SO}(3, \mathbb{R})$ (in particular it contains only elliptic elements) or it has a finite orbit (with one or two elements) on \mathbb{P}^1 [Bea, §5.1]. By definition, a *Kleinian group* is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

2.2. Holomorphic families of finitely generated subgroups of $\mathrm{PSL}(2, \mathbb{C})$. Let G be a finitely generated group and Λ be a connected complex manifold. A holomorphic family of representations of G into $\mathrm{PSL}(2, \mathbb{C})$ over Λ is a mapping

$$\rho : \Lambda \times G \longrightarrow \mathrm{PSL}(2, \mathbb{C}),$$

such that $\lambda \mapsto \rho_\lambda(g)$ is holomorphic for fixed g , and $g \mapsto \rho_\lambda(g)$ is a group homomorphism for fixed λ . We denote such a family by $(\rho_\lambda)_{\lambda \in \Lambda}$. For $g \in G$, we usually denote $\rho_\lambda(g)$ by g_λ .

Throughout the paper, we make the standing assumption that $(\rho_\lambda)_{\lambda \in \Lambda}$ is *generally faithful*, that is, that the set of parameters for which ρ_λ is not injective is a countable union of proper subvarieties. For this, it is enough that for some $\lambda_0 \in \Lambda$, ρ_{λ_0} is injective. We also assume that the family is non-trivial, that is, that the ρ_λ are not all conjugate in $\mathrm{PSL}(2, \mathbb{C})$.

Lemma 2.4. *If there exists a non-elementary representation in Λ , then the set of parameters $\lambda \in \Lambda$ for which $\rho_\lambda(G)$ is elementary is contained in a proper real analytic subvariety of Λ .*

Proof. As said before, there are two possibilities for $\rho_\lambda(G)$ to be elementary:

- [type I] either all elements have a common orbit of period 2, hence for every pair $f, g \in G$, $\mathrm{tr}[f_\lambda^2, g_\lambda^2] \equiv 2$ (Lemma 2.3),
- [type II] or all elements are elliptic, hence for every g_λ , $\mathrm{tr}^2(g_\lambda) \in [0, 4]$, and in particular $\Im \mathrm{tr}^2(g_\lambda) = 0$.

We see that the parameters for elementary subgroups satisfy a family of real analytic equations. \square

We say that a family of representations is *generally non-elementary* if it satisfies the assumption of the lemma. Since most of the problems that we consider are local, reducing the parameter space if necessary, it is not a restriction to assume that for all $\lambda \in \Lambda$, ρ_λ is non-elementary.

Two representations ρ_{λ_0} and ρ_{λ_1} are *quasi-conformally conjugate* if there exists a quasi-conformal homeomorphism $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ such that for every $g \in G$, $\rho_{\lambda_0}(g) \circ \phi = \phi \circ \rho_{\lambda_1}(g)$.

Definition 2.5. *We say that ρ_{λ_0} is stable if for λ close to λ_0 , ρ_λ is quasi-conformally conjugate to ρ_{λ_0} . The stability locus $\mathrm{Stab} \subset \Lambda$ is the open set of stable representations. Its complement is the bifurcation locus Bif .*

Theorem 2.6 (Sullivan [Su2], Bers [Ber]). *Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be a non-trivial, generally faithful, holomorphic family of non-elementary representations of G into $\mathrm{PSL}(2, \mathbb{C})$, and $\Omega \subset \Lambda$ be an open set. Then the following assertions are equivalent:*

- i. for every $\lambda \in \Omega$, $\rho_\lambda(G)$ is discrete;*
- ii. for every $\lambda \in \Omega$, ρ_λ is faithful;*
- iii. for every g in G , if for some $\lambda_0 \in \Omega$, g_{λ_0} is loxodromic (resp. parabolic, elliptic), then g_λ is loxodromic (resp. parabolic, elliptic) throughout Ω ;*
- iv. for any λ_0, λ_1 in Ω the representations ρ_{λ_0} and ρ_{λ_1} are quasi-conformally conjugate on \mathbb{P}^1 .*

In the situation of the theorem, due to *iii.*, the set of fixed points of $\rho_\lambda(g)$, $g \in G$ moves holomorphically. Furthermore, there exists a holomorphic motion of \mathbb{P}^1 , extending the motion of fixed points, and commuting with the action of G . The well-known Zassenhaus-Margulis lemma implies that the set DF of discrete and faithful representations is closed in parameter space (this is also referred to as Chuckrow's or Jørgensen's Theorem, see [Kap, p. 170]). We infer that the stability locus is the interior of DF. Observe in particular that, in contrast with rational dynamics, when non-empty the bifurcation locus has non-empty interior.

The previous theorem allows us to exhibit a dense (complex) codimension 1 phenomenon in the bifurcation locus. This is a basic source of motivation for the introduction of bifurcation currents.

Corollary 2.7. *For every $t \in [0, 4]$, the set of parameters λ_0 such that there exists $g \in G$ with $\mathrm{tr}^2 \rho_{\lambda_0}(g) = t$ and $\lambda \mapsto \mathrm{tr}^2 \rho_\lambda(g)$ is not constant at λ_0 , is dense in the bifurcation locus.*

Proof. For $t = 4$, the result is clear. For other values of t , consider an open set Ω with $\Omega \cap \mathrm{Bif} \neq \emptyset$. There exists $g \in G$ which changes type in Ω . Thus, the values of $\mathrm{tr}^2(g_\lambda)$, for $\lambda \in \Omega$, cross $[0, 4]$ along a non-empty open interval. We infer that for large k , the set of values of $\mathrm{tr}^2(g_\lambda^k)$, $\lambda \in \Lambda$, contains $[0, 4]$, hence the result. \square

In Section 4 we will require the notion of an *algebraic family of representations*. For this we need to introduce a few concepts; see e.g. [Kap] for a more detailed presentation. Fix a finite set $\{g_1, \dots, g_k\}$ generating G . The space $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ may be regarded as an algebraic subvariety V_G of $\mathrm{PSL}(2, \mathbb{C})^k$ by simply mapping a representation ρ to $(\rho(g_1), \dots, \rho(g_k)) \in \mathrm{PSL}(2, \mathbb{C})^k$, and observing that even if G is not finitely presented, V_G will be defined by finitely many equations. The same holds of course for $\mathrm{Hom}(G, \mathrm{SL}(2, \mathbb{C}))$. The algebraic structure of V_G is actually independent of the presentation of G . Notice also that V_G is defined over \mathbb{Q} .

There is an obvious embedding of $\mathrm{SL}(2, \mathbb{C})$ into \mathbb{C}^4 making $\mathrm{SL}(2, \mathbb{C})^k$ an affine subvariety of \mathbb{C}^{4k} . To view $\mathrm{PSL}(2, \mathbb{C})$ as an affine variety, observe that $\mathrm{PSL}(2, \mathbb{C})$ acts faithfully by conjugation on the space of 2-by-2 complex matrices of trace zero. This embeds $\mathrm{PSL}(2, \mathbb{C})$ into $\mathrm{GL}(3, \mathbb{C}) \subset \mathbb{C}^9$, and actually, $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to $\mathrm{SO}(3, \mathbb{C})$. We conclude that V_G is an affine algebraic variety (again, defined over \mathbb{Q}).

A holomorphic family of representations $(\rho_\lambda)_{\lambda \in \Lambda}$ is now simply a holomorphic mapping $\rho : \Lambda \rightarrow V_G \subset \mathbb{C}^{9k}$. We say that such a family is *algebraic* (resp. defined over K , where K is some subfield of \mathbb{C}) if there exists an algebraic subset $V \subset V_G$ (resp. defined over K) such that $\rho : \Lambda \rightarrow V$ is a dominant mapping. To say it differently, we require that the image $\rho(\Lambda)$ contains an open subset of an algebraic subset of V_G . This notion does not depend on the presentation of G .

2.3. Products of random matrices. In this paragraph we recall some classical facts on random matrix products that we specialize to our situation. The reader is referred to [BL, Furm] for more details and references. Fix a non-elementary representation $\rho : G \rightarrow \mathrm{PSL}(2, \mathbb{C})$. Let μ be a probability measure on the group G , whose support generates G as a semi-group. Let us first work under the following moment assumption:

$$(1) \quad \int_G \log \|\rho(g)\| d\mu(g) < \infty.$$

If the group G is finitely generated, and $\mathrm{length}(g)$ denotes the minimal length of a representation of g as a word in some fixed set of generators, then the condition

$$(2) \quad \int_G \mathrm{length}(g) d\mu(g) < \infty$$

clearly implies (1). An interesting case where these moment conditions (and also (7) and (8) below) are satisfied is that of the normalized counting measure on a finite symmetric set of generators of G .

The *Lyapunov exponent* of a representation $\rho : G \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is defined by the formula

$$(3) \quad \chi(\rho) := \lim_{n \rightarrow \infty} \frac{1}{n} \int_G \log \|\rho(g)\| d\mu^n(g),$$

where μ^n is the n^{th} convolution power of μ , that is the measure on G defined as the image of the product measure $\mu^{\otimes n}$ on G^n under the map $G^n \ni (g_1, \dots, g_n) \mapsto g_n \dots g_1 \in G$. Likewise, μ^n is the law of the n^{th} step of the left- or right- random walk on G with transition probabilities given by μ and starting at the identity.

Throughout the paper we use the following notation: $\mathbf{g} = (g_n)_{n \geq 1}$ denotes a sequence in $G^{\mathbb{N}}$, and $l_n(\mathbf{g}) = g_n \dots g_1$ (resp. $r_n(\mathbf{g}) = g_1 \dots g_n$) is the product on the left (resp. right) of the first n elements of \mathbf{g} .

The study of the Lyapunov exponent is closely related to that of the *transition operator*, that is, the Markov operator P acting on the space of continuous complex valued functions on \mathbb{P}^1 by $f \mapsto Pf$, where Pf is given by the formula

$$(4) \quad Pf(x) = \int_G f(\rho(g)x) d\mu(g).$$

A probability measure on \mathbb{P}^1 such that $\int Pf d\nu = \int f d\nu$ for all f is called *stationary*.

The following important result is due to Furstenberg [Furs].

Theorem 2.8. *Let ρ be a non-elementary representation of G and μ be a probability measure on G , generating G as a semi-group, and satisfying (1). Then the Lyapunov exponent defined in (3) is positive. Moreover, if $z_0 \in \mathbb{P}^1$ is fixed, then for $\mu^{\mathbb{N}}$ a.e. \mathbf{g} ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\rho(l_n(\mathbf{g}))Z_0\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\| = \chi(\rho),$$

where Z_0 denotes any lift of $z_0 \in \mathbb{P}^1$ to \mathbb{C}^2 .

Furthermore, there exists a unique stationary measure on \mathbb{P}^1 , which is diffuse and quasi-invariant under $\rho(G)$. Moreover, we have the formula

$$(5) \quad \chi = \int_{\mathbb{P}^1} \int_G \log \frac{\|\rho(g)(Z)\|}{\|Z\|} d\mu(g) d\nu(z)$$

(again Z is any lift of z).

The fact that for $\mu^{\mathbb{N}}$ -a.e. \mathbf{g} , $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\| = \chi(\rho)$ was originally proved by Furstenberg and Kesten [FK], and can nowadays easily be deduced from Kingman's Sub-additive Ergodic Theorem (see e.g. [Po]). Let us explain the argument. The measure $\mu^{\mathbb{N}}$ is invariant and ergodic under the shift $\sigma : (g_n) \mapsto (g_{n+1})$. Since, $\|\cdot\|$ is an operator norm, the family of functions $G^{\mathbb{N}} \ni \mathbf{g} \mapsto \log \|\rho(l_n(\mathbf{g}))\| \in \mathbb{R}^+$ defines a sub-additive cocycle, that is for every $m, n \geq 0$,

$$(6) \quad \log \|\rho(l_{n+m}(\mathbf{g}))\| \leq \log \|\rho(l_m(\mathbf{g}))\| + \log \|\rho(l_n(\sigma^n \mathbf{g}))\|.$$

By the sub-additive ergodic theorem, $\frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\|$ converges $\mu^{\mathbb{N}}$ -a.e. to a nonnegative number $\tilde{\chi}(\rho)$. Furthermore, by subadditivity we can write

$$0 \leq \frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\| \leq \frac{1}{n} \sum_{k=1}^n \log \|\rho(l_1(\sigma^k(\mathbf{g})))\|,$$

where by the Birkhoff ergodic theorem and (1), the right hand side converges in $L^1(\mu^{\mathbb{N}})$. This domination implies that $\frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\|$ also converges in L^1 . By integrating against $\mu^{\mathbb{N}}$, we therefore conclude that $\tilde{\chi}(\rho)$ must equal $\chi(\rho)$, and the result follows.

We note for future reference that when ρ is non-elementary, the support of the stationary measure coincides with the *limit set* of the representation ρ , defined as the minimal closed $\rho(G)$ -invariant subset of the Riemann sphere. The proof goes as follows: since ν is quasi-invariant under $\rho(G)$, $\text{Supp}(\nu)$ is closed and $\rho(G)$ -invariant, thus it must contain the limit set. Conversely, by uniqueness of the stationary measure, the limit set cannot be a proper subset of $\text{Supp}(\nu)$.

We now discuss the exponential convergence of the iterates of the transition operator to the stationary measure, due to Le Page. For this, denote by N the operator of integration against the stationary measure ν ,

$$N : f \mapsto \int f d\nu.$$

Let C^α be the space of Hölder continuous functions on \mathbb{P}^1 endowed with the norm

$$\|f\|_{C^\alpha} = \|f\|_\infty + m_\alpha(f), \text{ with } m_\alpha(f) := \sup_{x \neq y \in \mathbb{P}^1} \left(\frac{|f(x) - f(y)|}{d_{\mathbb{P}^1}(x, y)} \right)^\alpha,$$

($d_{\mathbb{P}^1}$ is the spherical distance).

We also need stronger moment assumptions on μ : we assume that (G, μ, ρ) is non-elementary and satisfies the following:

$$(7) \quad \text{there exists } \tau > 0 \text{ such that } \int_G \|\rho(g)\|^\tau d\mu < \infty.$$

As above, it is enough that μ satisfies an exponential moment condition in G :

$$(8) \quad \text{there exists } \sigma > 0 \text{ such that } \int_G \exp(\sigma \text{length}(g)) d\mu < \infty.$$

The following important result is due to Le Page [L1].

Theorem 2.9. *Let (G, μ, ρ) be a non-elementary representation satisfying (7). Then there exists $\alpha, \beta > 0$, and a constant C such that for every $n \geq 0$*

$$(9) \quad \|P^n - N\|_{C^\alpha} \leq C e^{-\beta n}.$$

This in turn follows from an estimate on average contraction: there exists $0 < \alpha < \tau$ and an integer n_0 such that

$$(10) \quad \sup_{x \neq y \in \mathbb{P}^1} \int \left(\frac{d_{\mathbb{P}^1}(\rho(g)x, \rho(g)y)}{d_{\mathbb{P}^1}(x, y)} \right)^\alpha d\mu^{n_0}(g) < 1.$$

Important consequences of these estimates are versions for random matrix products of the classical limit theorems for i.i.d. random variables: Central Limit Theorem, Large Deviations Theorem, etc.

Another result, due to Guivarc'h, will be useful to us.

Theorem 2.10 ([Gu, Theorem 9]). *Let (G, μ, ρ) be a non-elementary representation satisfying (7). Then for $\mu^{\mathbb{N}}$ a.e. \mathbf{g} we have that*

$$(11) \quad \frac{1}{n} \log |\mathrm{tr}(\rho(l_n(\mathbf{g})))| = \frac{1}{n} \log |\mathrm{tr}(\rho(g_n \cdots g_1))| \xrightarrow{n \rightarrow \infty} \chi(\rho).$$

We actually give a proof of a refined version of this theorem in Appendix A below. The following corollary is immediate.

Corollary 2.11. *Let (G, μ, ρ) be as in Theorem 2.10. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be bounded from below and equivalent to $\log x$ when $x \rightarrow +\infty$. Then*

$$\frac{1}{n} \int h(|\mathrm{tr}(\rho(g))|) d\mu^n(g) \xrightarrow{n \rightarrow \infty} \chi(\rho).$$

For instance, if $\lambda_{\max}(\rho(g))$ denotes the spectral radius of $\rho(g)$, then we have that

$$\frac{1}{n} \int \log |\lambda_{\max}(\rho(g))| d\mu^n(g) \rightarrow \chi(\rho).$$

On the other hand one cannot expect in general to have convergence in $L^1(G, \mu^{\mathbb{N}})$ in (11). Indeed, if $\rho(G)$ contains the rotation of angle π , whose trace is 0, then $\frac{1}{n} \int \log |\mathrm{tr}(\rho(g))| d\mu^n(g)$ takes the value $-\infty$ infinitely often. To get a less trivial example, if $\rho(G)$ contains a rotation with well chosen angle (i.e. with many iterates very close to angle π) then the sequence $\frac{1}{n} \int \log |\mathrm{tr}(\rho(g))| d\mu^n(g)$ may admit cluster values smaller than $\chi(\rho)$. The same phenomenon of course occurs when considering $\log |\mathrm{tr}^2(\rho(g)) - t|$ for some $t \in [0, 4]$.

Proof of the corollary. By Theorem 2.10, $\frac{1}{n} h(|\mathrm{tr}(\rho(l_n(\mathbf{g})))|) \rightarrow \chi(\rho)$ a.s. Furthermore there exists a constant C such that $-C \leq h(|\mathrm{tr}(\rho(g))|) \leq \max(C \log \|\rho(g)\|, C)$, so the result follows from the Dominated Convergence Theorem. \square

Remark 2.12. In the sequel, we often need some uniformity on the estimates (9) with respect to ρ . To see why such a uniformity is true, it is instructive to recall how (10) implies (9). If we set $c_n := \sup_{x \neq y \in \mathbb{P}^1} \int \left(\frac{d_{\mathbb{P}^1}(\rho(g)x, \rho(g)y)}{d_{\mathbb{P}^1}(x, y)} \right)^\alpha d\mu^n(g)$, then for any function f in C^α ,

$$m_\alpha(P^n f) \leq c_n \cdot m_\alpha(f).$$

Since ν is stationary, we infer that $\|P^n - N\|_{C^\alpha} \leq c_n$. Now, it is straightforward to check that $c_{m+n} \leq c_n c_m$ for every pair of integers m, n . Furthermore, under the condition (10), we get that for every integer n , $c_n \leq C \cdot e^{-\beta n}$ with $\beta = -\frac{1}{n_0} \log c_{n_0} > 0$ and $C = \sup_{k < n_0} c_k$, and thus the estimate (9) holds. A useful consequence of this is that the constants C , α , and β in (9) can be chosen uniformly in a neighborhood of ρ , since under our moment assumption c_n depends continuously on ρ .

3. THE BIFURCATION CURRENT

Throughout this section we fix a holomorphic family of representations $(\rho_\lambda)_{\lambda \in \Lambda}$ of G into $\mathrm{PSL}(2, \mathbb{C})$, that we assume to be generally faithful, non-trivial, and for all λ , non-elementary. We fix a probability measure μ on G , generating G as a semi-group, and satisfying (2). In particular (1) holds, locally uniformly in λ . From now on, such families of representations (endowed with a measure μ on G) will be called *admissible*.

For basics on plurisubharmonic (psh for short) functions and positive currents, the reader is referred to [H2, De]. Recall that a positive closed current T of bidegree $(1,1)$ locally admits a psh potential u , i.e. $T = dd^c u$ and that it is possible to pull-back such currents under holomorphic maps by pulling back the potentials. Another frequently used result is the so-called Hartogs' lemma [H2, pp. 149-151] which asserts that families of psh functions with uniform bounds from above have good compactness properties.

3.1. Definition. Given an admissible family of representations (G, μ, ρ) as above, we let $\chi(\lambda) := \chi(\rho_\lambda)$ be the Lyapunov exponent of ρ_λ .

Proposition 3.1. *The Lyapunov exponent χ defines a continuous psh function on Λ , which is pluriharmonic on the stability locus.*

Proof. For $g \in G$, $\lambda \mapsto \log \|g_\lambda\|$ is the supremum of the family of psh functions $\lambda \mapsto \log \|g_\lambda Z_0\|$, where Z_0 ranges over the unit sphere in \mathbb{C}^2 , and it is continuous because the norm is. We thus infer from [H2, Thm 4.1.2] that $\lambda \mapsto \log \|g_\lambda\|$ is psh. Hence χ is psh since by (3), it is the pointwise limit of a uniformly bounded sequence of psh functions.

Another proof goes by observing that we can replace $\|\cdot\|$ by $\|\cdot\|_2$ in the definition of χ , in which case its plurisubharmonicity is obvious.

The continuity of χ is a consequence of Furstenberg's formula together with the fact that the stationary measure is unique, and therefore depends continuously on λ in the weak topology (see [Furm, §1.13]).

Finally, the second assertion of Corollary 2.11 implies that χ is pluriharmonic on the stability locus. Indeed, locally the g_λ do not change type there, so the multipliers of fixed points vary holomorphically, without crossing the unit circle (by possibly staying constant of modulus 1), and we infer that χ is a limit of pluriharmonic functions, hence itself pluriharmonic. \square

Definition 3.2. *If (G, μ, ρ) is an admissible family of representations of G into $\mathrm{PSL}(2, \mathbb{C})$, the bifurcation current T_{bif} is defined by $T_{\mathrm{bif}} = dd^c \chi$.*

Proposition 3.1 implies that the support of T_{bif} is contained in the bifurcation locus.

Remark 3.3. The Lyapunov exponent of an elementary representation is well defined by the formula (3). Thus, if the subset of non-elementary representations is not empty, χ still defines a locally bounded psh function on Λ , and it makes perfect sense to talk about the bifurcation current also in this case.

We close this subsection by studying the regularity of the bifurcation current. The continuity of χ will be a technically useful fact in the paper. For an admissible family satisfying an exponential moment condition, it was shown by Le Page in [L2] that χ is actually Hölder continuous. For the reader's convenience, we give a short proof of this result in the case where μ has finite support. Notice that the key argument is the exponential convergence of the transition operator (Theorem 2.9).

Theorem 3.4 (Le Page). *Let (G, μ, ρ) be an admissible family of representations, satisfying (7) locally uniformly in λ (e.g. satisfying (8)). Then the Lyapunov exponent function is Hölder continuous.*

Proof in the finite support case. Fix a parameter λ_0 , and some small neighborhood $U \ni \lambda_0$, that we view through a chart as an open set in $\mathbb{C}^{\dim \Lambda}$. We also consider a norm in $\mathbb{C}^{\dim \Lambda}$ that we simply denote by $|\cdot|$. For $\lambda, \lambda' \in U$, write $\rho_{\lambda'}(g) = \rho_\lambda(g) + \varepsilon(g)$. Since $\text{Supp}(\mu)$ is finite, when $g \in \text{Supp}(\mu)$, $\varepsilon(g) = O(|\lambda' - \lambda|)$. For notational simplicity let us denote $g' = \rho_{\lambda'}(g)$ and $g = \rho_\lambda(g)$. For an integer n (that will be chosen of the order of magnitude of $-\log |\lambda' - \lambda|$), we have

$$\begin{aligned} (g'_n \dots g'_1)(g_n \dots g_1)^{-1} &= (g_n + \varepsilon_n) \dots (g_1 + \varepsilon_1)(g_1^{-1} \dots g_n^{-1}) \\ &= (I + \varepsilon_n g_n^{-1})(I + \text{Ad}(g_n)(\varepsilon_{n-1} g_{n-1}^{-1})) \dots (I + \text{Ad}(g_n \dots g_2)(\varepsilon_1 g_1^{-1})). \end{aligned}$$

We want to estimate the distance between the latter matrix and the identity. First observe that there exists a constant $M > 1$ (the square of the maximum of the norms of the elements $\rho_\lambda(g)$ with $\lambda \in U$ and $g \in \text{Supp}(\mu)$) such that

$$(12) \quad \|\text{Ad}(g_n \dots g_{k+1})(\varepsilon_k g_k^{-1})\| \leq M^{n-k} |\lambda' - \lambda|.$$

For $1 \leq k \leq n$, let $u_k = \|\text{Ad}(g_n \dots g_{k+1})(\varepsilon_k g_k^{-1})\|$ and $v = (I + u_1) \dots (I + u_n) - I$. Since $k \leq n$, and $M > 1$, (12) shows that $\|u_k\| \leq M^n |\lambda - \lambda'|$. Expanding v we obtain that

$$\|v\| = \left\| \sum_{0 < k \leq n} \sum_{i_1 < \dots < i_k} u_{i_1} \dots u_{i_k} \right\| \leq \sum_{0 < k \leq n} C_n^k M^{kn} |\lambda - \lambda'|^k = (1 + x_n)^n - 1 \leq n x_n (1 + x_n)^{n-1},$$

where $x_n = M^n |\lambda - \lambda'|$. Now choose n so that $n x_n = |\lambda - \lambda'|^{1/2}$, that is, $n M^n = |\lambda - \lambda'|^{-1/2}$. The quantity $(1 + x_n)^{n-1}$ is bounded since $n x_n$ is constant. Thus we get that

$$\|v\| = \|(g'_n \dots g'_1)(g_n \dots g_1)^{-1} - I\| \leq C^{\text{st}} |\lambda - \lambda'|^{1/2}.$$

Note that there is a constant C^{st} such that for every $g \in \text{PSL}(2, \mathbb{C})$ and every $y \in \mathbb{P}^1$, we have $d_{\mathbb{P}^1}(g(y), y) \leq C^{\text{st}} \cdot \|g - I\|$. Thus, for every $x \in \mathbb{P}^1$, we have that

$$d_{\mathbb{P}^1}((g'_n \dots g'_1)x, (g_n \dots g_1)x) = d_{\mathbb{P}^1}((g'_n \dots g'_1)(g_n \dots g_1)^{-1}y, y) \leq C^{\text{st}} |\lambda' - \lambda|^{1/2},$$

by denoting $y = g_n \dots g_1(x)$. As a consequence, if f is a Hölder continuous function of exponent α ,

$$|P_{\lambda'}^n f(x) - P_\lambda^n f(x)| \leq C^{\text{st}} |\lambda' - \lambda|^{\alpha/2} \|f\|_{C^\alpha}.$$

We actually need to apply the latter estimate for a function which also depends on λ , but in a differentiable way, namely,

$$f_\lambda(x) = \int \log \frac{\|\rho_\lambda(g)X\|}{\|X\|} d\mu(g) \quad (X \text{ a lift of } x).$$

There exists some constant for which in the given neighborhood of λ_0 , we have $\|f_{\lambda'} - f_\lambda\|_\infty \leq C^{\text{st}} |\lambda' - \lambda|$. We can thus write $\|P_{\lambda'}^n f_{\lambda'} - P_\lambda^n f_\lambda\|_\infty \leq \|P_{\lambda'}^n (f_{\lambda'} - f_\lambda)\|_\infty + \|(P_{\lambda'}^n - P_\lambda^n) f_\lambda\|_\infty$ and consequently

$$(13) \quad \|P_{\lambda'}^n f_{\lambda'} - P_\lambda^n f_\lambda\|_\infty \leq C^{\text{st}} |\lambda' - \lambda|^{\alpha/2}.$$

To finish the proof, notice that our choice of n implies that

$$n = \frac{-\log |\lambda - \lambda'|}{2 \log M} + O(\log |\log |\lambda - \lambda'|||) \sim \frac{-\log |\lambda - \lambda'|}{2 \log M}.$$

Therefore, by the exponential convergence (9) of P_λ^n towards N_λ , we obtain that for $\lambda \in U$,

$$(14) \quad \|(P_\lambda^n - N_\lambda)f_\lambda\|_\infty \leq C^{\text{st}}|\lambda' - \lambda|^\gamma$$

for any $\gamma < \frac{\beta}{2\log(M)}$ (recall from Remark 2.12 that β is locally uniform). By Furstenberg's formula (5), $N_\lambda f_\lambda = \chi_\lambda$, so we conclude by summing (13) and (14) that χ is Hölder continuous in U , of exponent γ , for any $\gamma < \min(\alpha/2, \frac{\beta}{2\log(M)})$. \square

3.2. Geometric interpretation. We now consider the fibered action of G on $\Lambda \times \mathbb{P}^1$, that is, for $g \in G$, we define \widehat{g} by $\widehat{g} : (\lambda, z) \mapsto (\lambda, g_\lambda(z))$. If $p \in \mathbb{P}^1$, we let $\widehat{g} \cdot p := \{(\lambda, g_\lambda(p)), \lambda \in \Lambda\}$. More generally, objects living in $\Lambda \times \mathbb{P}^1$ are marked with a hat. Let also π_1 and π_2 be the respective coordinate projections from $\Lambda \times \mathbb{P}^1$ to Λ and \mathbb{P}^1 .

The following theorem gives a geometric characterization of the bifurcation current.

Theorem 3.5. *Let (G, μ, ρ) be an admissible family of representations of G into $\text{PSL}(2, \mathbb{C})$. Fix $z_0 \in \mathbb{P}^1$, and for every n define a current of bidegree $(1, 1)$ on $\Lambda \times \mathbb{P}^1$ by the formula*

$$\widehat{T}_n = \frac{1}{n} \int [\widehat{g} \cdot z_0] d\mu^n(g).$$

Then the sequence (\widehat{T}_n) converges to $\pi_1^(T_{\text{bif}})$.*

This implies that $\lambda_0 \in \text{Supp}(T_{\text{bif}})$ if and only if for every neighborhood $U \ni \lambda_0$, the average volume of $\widehat{g \cdot p_0} \cap \pi_1^{-1}(U)$, relative to μ^n , grows linearly in n . If U is contained in the stability locus, it is easy to show that $\bigcup_{g \in G} \widehat{g \cdot p_0} \cap \pi_1^{-1}(U)$ is a normal family of graphs over any relatively compact subset of U , hence $\widehat{T}_n \rightarrow 0$ in $\pi_1^{-1}(U)$. We thus obtain an alternate proof of the fact that $\text{Supp}(T_{\text{bif}}) \subset \text{Bif}$.

Proof. This is a local result on Λ , so we may assume that Λ is a ball in \mathbb{C}^k , endowed with its standard Kähler form ω . Let also $\omega_{\mathbb{P}^1}$ be the Fubini-Study form on \mathbb{P}^1 associated to our choice of Hermitian norm. On $\Lambda \times \mathbb{P}^1$ we choose the Kähler form $\widehat{\omega} := \pi_1^*\omega + \pi_2^*\omega_{\mathbb{P}^1}$.

The first observation is that $\langle \widehat{T}_n, \pi_1^*\omega^k \rangle \rightarrow 0$. Indeed, for $g \in G$, since $\widehat{g} \cdot z_0$ is a graph, $[\widehat{g} \cdot z_0] \wedge \pi^*\omega^k = \int_\Lambda \omega^k$, so $\langle \widehat{T}_n, \pi_1^*\omega \rangle = O(\frac{1}{n})$. Thus, if we can show that (\widehat{T}_n) has locally uniformly bounded mass, every cluster value \widehat{T} of this sequence satisfies $\widehat{T} \wedge \pi^*\omega^k = 0$. In this case it is classical that \widehat{T} does not depend on the \mathbb{P}^1 coordinate, in the sense that there exists a current T on Λ such that $\widehat{T} = \pi_1^*T$. For completeness we sketch a proof of this fact in Lemma 3.7 below. Now, $\widehat{\omega}^k$ is equal to $\pi_1^*\omega^k + k\pi_1^*\omega^{k-1} \wedge \pi_2^*\omega_{\mathbb{P}^1}$, therefore, since $\langle \widehat{T}_n, \pi_1^*\omega^k \rangle \rightarrow 0$ we are led to understand pairings of the form $\langle \widehat{T}_n, \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle$, where ϕ is a $(k-1, k-1)$ test form on Λ , or equivalently, to understand $(\pi_1)_*(\widehat{T}_n \wedge \pi_2^*\omega_{\mathbb{P}^1})$.

For this we compute

$$(15) \quad \begin{aligned} \langle \widehat{T}_n, \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle &= \frac{1}{n} \int \langle [\widehat{g} \cdot z_0], \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle d\mu^n(g) \\ &= \frac{1}{n} \int \left(\int_\Lambda (\pi_1|_{\widehat{g} \cdot z_0})_* (\pi_2^*\omega_{\mathbb{P}^1}|_{\widehat{g} \cdot z_0}) \wedge \phi \right) d\mu^n(g) \\ &= \frac{1}{n} \int \left(\int_\Lambda (\pi_2 \circ (\pi_1|_{\widehat{g} \cdot z_0})^{-1})^* \omega_{\mathbb{P}^1} \wedge \phi \right) d\mu^n(g), \end{aligned}$$

where in the second line we use the fact that for every g , $\pi_1|_{\widehat{g} \cdot z_0}$ is a biholomorphism. Now observe that for $g \in G$, the map $\pi_2 \circ (\pi_1|_{\widehat{g} \cdot z_0})^{-1}$ is just defined by the formula $\lambda \mapsto g_\lambda(z_0)$.

Denote it by h . We have that $h^*\omega_{\mathbb{P}^1} = dd^c \log \|H\|$, where $H : \Lambda \rightarrow \mathbb{C}^2 \setminus \{0\}$ is any lift of h . Denote $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as before, $z_0 = [x_0 : y_0]$, and $Z_0 = (x_0, y_0)$ be a lift of z_0 to \mathbb{C}^2 . We infer that

$$h^*\omega_{\mathbb{P}^1} = dd^c \log \left(|a_\lambda x_0 + b_\lambda y_0|^2 + |c_\lambda x_0 + d_\lambda y_0|^2 \right)^{\frac{1}{2}} = dd^c \log \|g_\lambda(Z_0)\| = dd^c \log \frac{\|g_\lambda(Z_0)\|}{\|Z_0\|},$$

where the dd^c takes place in the λ variable. We conclude that

$$(16) \quad \langle \widehat{T}_n, \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle = \int_\Lambda \phi \wedge dd^c_\lambda \left(\frac{1}{n} \int \log \frac{\|g_\lambda(Z_0)\|}{\|Z_0\|} d\mu^n(g) \right).$$

By Theorem 2.8 for every $z_0 \in \mathbb{P}^1$ and every λ ,

$$\frac{1}{n} \int \log \frac{\|g_\lambda(Z_0)\|}{\|Z_0\|} d\mu^n(g) \xrightarrow{n \rightarrow \infty} \chi(\lambda).$$

Furthermore by the subadditivity of $\|g_\lambda\|$ and the uniform moment condition (2), this sequence is locally uniformly bounded above (with respect to λ), hence by the Hartogs Lemma the convergence holds in L^1_{loc} and we finally obtain that $\lim_n (\pi_1)_*(\widehat{T}_n \wedge \pi_2^*\omega_{\mathbb{P}^1}) = dd^c \chi = T_{\text{bif}}$.

Applying this to $\phi = \varphi \omega^{k-1}$, where φ is a cutoff function, we see that the sequence (\widehat{T}_n) has locally uniformly bounded mass. Let \widehat{T} be any of its cluster values. We know that it is of the form π_1^*T and that $(\pi_1)_*(\widehat{T} \wedge \pi_2^*\omega_{\mathbb{P}^1}) = T_{\text{bif}}$. The following classical computation then finishes the proof. \square

Lemma 3.6. *With notation as above, $(\pi_1)_*(\pi_1^*T \wedge \pi_2^*\omega_{\mathbb{P}^1}) = T$*

Proof. Let ϕ be a test $(k-1, k-1)$ form in Λ . We have that

$$\langle (\pi_1)_*(\pi_1^*T \wedge \pi_2^*\omega_{\mathbb{P}^1}), \phi \rangle = \int \pi_1^*(T \wedge \phi) \wedge \pi_2^*\omega_{\mathbb{P}^1}.$$

Now, $T \wedge \phi$ is a current of bidegree (k, k) with compact support in Λ , so it is cohomologous (with compact supports) to $(\int T \wedge \phi) \Theta$, where Θ is any compactly supported positive smooth (k, k) form of integral 1. So we deduce that

$$\int \pi_1^*(T \wedge \phi) \wedge \pi_2^*\omega_{\mathbb{P}^1} = \left(\int T \wedge \phi \right) \int \pi_1^*\Theta \wedge \pi_2^*\omega_{\mathbb{P}^1} = \int T \wedge \phi,$$

and we are done. \square

As promised above, we sketch the proof of the following classical fact.

Lemma 3.7. *Let $B_1 \times B_2 \subset \mathbb{C}^k \times \mathbb{C}$ be a product of balls. Write $z = (z', z_{k+1})$ for the coordinate on \mathbb{C}^{k+1} and denote by ω_1 the standard Kähler form on \mathbb{C}^k . Let T be a positive closed current of bidegree $(1, 1)$ on $B_1 \times B_2$, such that $T \wedge \omega_1^k = 0$. Then there exists a positive closed current T_1 on B_1 such that $T = \pi_1^*T_1$, π_1 being the first projection.*

Proof. Decompose T in coordinates as $T = i \sum T_{i,j} dz_i \wedge d\bar{z}_j$, where $(T_{i,j})$ is a Hermitian matrix of measures. Since $T \wedge \omega_1^k = 0$, $T_{k+1, k+1} = 0$. Positivity implies that $T_{k+1, j} = 0$ for all j (see [De, Prop. 1.14]). Then closedness implies that the $T_{i,j}$, $i, j \leq k$ do not depend on z_{k+1} (see the proof of Theorem 2.13 in [De]). The result is proved. \square

In the next –presumably well-known– proposition we give an estimate for the speed of convergence of the potentials appearing in the proof of Theorem 3.5. This will play a crucial role in Theorem 3.9.

Proposition 3.8. *Let (G, μ, ρ) be an admissible family of representations of G into $\mathrm{PSL}(2, \mathbb{C})$, satisfying the exponential moment condition (8). Then for every $z_0 \in \mathbb{P}^1$ we have that*

$$(17) \quad \frac{1}{n} \int \log \frac{\|g\lambda(Z_0)\|}{\|Z_0\|} d\mu^n(g) = \chi(\lambda) + O\left(\frac{1}{n}\right),$$

where the $O(\cdot)$ is locally uniform in λ .

Proof. Fix λ for the moment and let f be the function on \mathbb{P}^1 defined by $f(z) = \int \log \frac{\|g(Z)\|}{\|Z\|} d\mu(g)$ (we drop the λ from the formulas). Recall Furstenberg's formula that $\int f d\nu = \chi$, where ν is the unique stationary measure. We have that

$$(18) \quad \begin{aligned} \int \log \frac{\|g(Z_0)\|}{\|Z_0\|} d\mu^n(g) - n\chi &= \sum_{k=1}^n \left(\int \log \frac{\|gg_{k-1} \cdots g_1(Z_0)\|}{\|g_{k-1} \cdots g_1(Z_0)\|} d\mu(g) d\mu(g_{k-1}) \cdots d\mu(g_1) - \chi \right) \\ &= \sum_{k=1}^n \left(P^{k-1}(f)(z_0) - \int f d\nu \right), \end{aligned}$$

furthermore, under the moment condition (7), we know that there exist constants $C > 0$, and $\beta < 1$ such that $\|P^k(f) - \int f d\nu\|_{L^\infty} < C\beta^k$. We thus conclude that the sum in (18) is bounded as $n \rightarrow \infty$, yielding the desired estimate for fixed λ . For the uniformity statement, just recall from Remark 2.12 that the values of C and β are locally uniform in λ . \square

3.3. The support of T_{bif} . We keep hypotheses as before, keeping in particular from the last proposition the exponential moment condition (8) (it would actually be enough to assume that (7) holds locally uniformly).

Here is the precise statement of the characterization of the support of T_{bif} .

Theorem 3.9. *Let (G, μ, ρ) be an admissible family of representations of G into $\mathrm{PSL}(2, \mathbb{C})$, satisfying the exponential moment condition (8).*

Then the support of T_{bif} coincides with the bifurcation locus.

We know from Proposition 3.1 that $\mathrm{Supp}(T_{\mathrm{bif}}) \subset \mathrm{Bif}$ so only the reverse inclusion needs to be established. For this, we make use of the geometric interpretation of T_{bif} given in §3.2.

Since it will play a very important role in the proof, let us start by reviewing the construction of the Poisson boundary of (G, μ) (see [Kai] for more details). Consider the right random walk on the group G , defined as the Markov chain on G with transition probabilities given by $p(x, y) = \mu(x^{-1}y)$. As a measurable space, the Poisson boundary is the set of paths $(r_n) \in G^{\mathbb{N}}$, equipped with the tail algebra \mathcal{T} , that is the algebra of Borel sets in $G^{\mathbb{N}}$ which are invariant by the shift $\sigma(r_n) = (r_{n+1})$. Hence, two paths (r_n) and (r'_n) have to be considered as equivalent in the Poisson boundary as soon as they have the same tails. We denote the Poisson boundary by $P(G, \mu)$.

It inherits a measure class induced by the μ -random walk on G , as follows. Recall that the position r_n ($n \geq 1$) of the random walk at time n is deduced from its position at time 0 by the following formula

$$(19) \quad r_n = r_0 h_1 \cdots h_n,$$

where the h_i are mutually independant random variables with distribution μ . Any initial distribution θ on G determines a Markov measure \mathbf{P}_θ on the space of paths $(r_n) \in G^{\mathbb{N}}$: the image of $\theta \otimes \mu^{\mathbb{N}}$ under the assignment $(r_0, (h_i)) \mapsto (r_n)$ given in (19). We shall denote by

P_g the Markov measure corresponding to the Dirac mass at the point g . It is straightforward to verify that P_e is μ -stationary, i.e. $P_e = \int P_g d\mu(g)$. Note also that the measures P_g are absolutely continuous with respect to each other.

The coordinate-wise left multiplications by an element of the group G on $G^{\mathbb{N}}$ commutes with the shift and induces an action of G on $P(G, \mu)$. The measure P_e is pushed by an element g of G on the measure P_g , so that the measure P_e is quasi-preserved by this action.

Proof of Theorem 3.9. It is no loss of generality to assume that $\dim(\Lambda) = 1$. Let $V \subset \Lambda$ be an open subset where T_{bif} vanishes, and $U \Subset V$. We want to show that U is contained in the stability locus. Fix $z_0 \in \mathbb{P}^1$ and define \hat{T}_n as in Theorem 3.5. More generally we keep notation as in the proof of that theorem.

Step 1. The vanishing of T_{bif} on U implies the mass estimate $\int_{\pi_1^{-1}(U)} \hat{T}_n \wedge \hat{\omega} = O\left(\frac{1}{n}\right)$.

Indeed, recall that $\hat{\omega} = \pi_1^* \omega + \pi_2^* \omega_{\mathbb{P}^1}$. We already observed that $\langle \hat{T}_n, \pi_1^* \omega \rangle = O\left(\frac{1}{n}\right)$. For the second term, the computations in the proof of Theorem 3.5 show that

$$\begin{aligned} \langle \hat{T}_n|_{\pi_1^{-1}(U)}, \pi_2^* \omega_{\mathbb{P}^1} \rangle &= \int_U dd^c \chi_n, \text{ where } \chi_n = \frac{1}{n} \int \log \frac{\|g_\lambda(Z_0)\|}{\|Z_0\|} d\mu^n(g) \\ &= \int_U dd^c (\chi_n - \chi), \text{ because } \chi \text{ is harmonic on } U. \end{aligned}$$

Since by Proposition 3.8, $\|\chi_n - \chi\|_{L^\infty} = O\left(\frac{1}{n}\right)$, introducing a cut-off function and integrating by parts shows that the last integral is $O\left(\frac{1}{n}\right)$, which was the result to be proved. We note that a similar argument appears in [DF, Thm 3.2].

Step 2. Construction of a holomorphic equivariant map from the Poisson boundary (a partially defined map is extended using Bishop's theorem and the mass estimate).

Lemma 3.10. *There exists a measurable map $\theta : P(G, \mu) \times U \rightarrow \mathbb{P}^1$, defined almost everywhere with respect to the first factor, which is holomorphic with respect to the second variable, and G -equivariant with respect to the first.*

To prove the lemma, using notation as in the paragraph preceding the proof of the theorem, we will define the map θ on $\Omega \times U$ for a measurable subset $\Omega \subset \{e\} \times G^{\mathbb{N}}$ of full P_e measure, and then we will extend it on the union $\bigcup_{g \in G} g\Omega \times U$ by the formula $\theta(r, \lambda) = \rho(r_0)_\lambda \cdot \theta(r_0^{-1}r)$. To verify that the extension is G -equivariant and tail invariant –thus defining the desired equivariant map on the Poisson boundary– it will be sufficient to check that θ depends only on the tail of the first variable, and satisfies

$$(20) \quad \theta(r, \lambda) = \rho(r_1)_\lambda \cdot \theta(r_1^{-1}\sigma(r), \lambda),$$

for every $r \in \Omega$.

Proof of Lemma 3.10. Recall from [BL, Corollary 7.1, p. 40] that if λ is fixed, a map as in (20) exists and is unique. More precisely, for P_e -a.e. $r \in G^{\mathbb{N}}$, the sequence $r_n(z_0)$ converges to a point z_r , independent of z_0 , and the law of z_r is ν . So if we fix a dense sequence (λ_q) in U , we obtain a set $\Omega_1 \subset G^{\mathbb{N}}$ of full P_e -measure such that for every $r \in \Omega_1$, and every q , $r_{n, \lambda_q}(z_0)$ converges to some z_{r, λ_q} . The proof of lemma 3.10 consists in showing that we can interpolate this function of λ_q by a holomorphic function of λ .

We claim that there exists $\Omega_2 \subset G^{\mathbb{N}}$ of full P_e -measure such that for $r \in \Omega_2$ there exists a subsequence n_j such that $\text{vol}(\hat{r}_{n_j} \cdot z_0)$ is bounded. Indeed, since the random walk is conditioned to start at $r_0 = e$, the distribution of r_n is μ^n , hence from Step 1 we deduce that there is a constant C such that for every n ,

$$\int \text{vol}(\hat{r}_n \cdot z_0) dP_e(r) \leq C.$$

Our claim now follows from the following elementary argument: for $r \in G^{\mathbb{N}}$, let $\varphi_n(r) = \text{vol}(\hat{r}_n \cdot z_0)$ and $\psi_n(r) = \inf_{k \geq n} \varphi_k(r)$. The sequence ψ_n is increasing, so

$$C \geq \limsup \int \varphi_n dP_e \geq \lim \int \psi_n dP_e = \int \lim \psi_n dP_e,$$

and we conclude that $\lim \psi_n$ is a.s. finite, which was the desired result.

Recall that if $f_n : U \rightarrow \mathbb{P}^1$ is a sequence of holomorphic mappings such that the corresponding graphs have uniformly bounded volume, then by Bishop's Theorem [Ch, §15.5] it admits a convergent subsequence, up to finitely many “vertical bubbles”, that is, there exists a subsequence n_j , $f : U \rightarrow \mathbb{P}^1$ and a finite subset E in U such that f_{n_j} converges to f uniformly on compact subsets of $U \setminus E$.

Now by putting $\Omega = \Omega_1 \cap \Omega_2$, we are able to construct the mapping θ . Indeed if $r \in \Omega$, there exists a subsequence n_j such that the sequence of graphs $\hat{r}_{n_j} \cdot z_0$ has bounded volume. Extracting again if necessary we may assume that it converges, up to possibly finitely many bubbles. Let $f : U \rightarrow \mathbb{P}^1$ be the limit. Then for all but possibly finitely many λ_q (where bubbling occurs), we have that $f(\lambda_q) = z_{r, \lambda_q}$. Thus the assignment $\lambda_q \mapsto z_{r, \lambda_q}$ admits a (necessary unique since (λ_q) is dense) continuation as a holomorphic mapping $U \rightarrow \mathbb{P}^1$. Likewise, f is the only possible cluster value of $\hat{r}_n \cdot z_0$ since it is determined by the z_{r, λ_q} . We can thus define $\theta(r, \lambda)$ to be $f(\lambda)$. The same argument shows that this function depends only on the tail of r , and satisfies (20). \square

It is straightforward that the image of P_e under an equivariant map is stationary. From this we get the following statement, which will be used at several places below.

Lemma 3.11. *Let θ be the mapping constructed in Lemma 3.10. Then $(\theta(\cdot, \lambda))_* P_e$ is the stationary measure ν_λ on \mathbb{P}^1 .*

Remark 3.12. This argument shows that the estimate in Proposition 3.8 cannot be substantially improved. Indeed, assume on the contrary that the $O(\frac{1}{n})$ in (17) can be replaced by $o(\frac{1}{n})$. Then we infer that over the stability locus, the average projected volume of $\hat{r}_n \cdot z_0$ on the second factor (i.e. \mathbb{P}^1) tends to zero. Therefore, the limiting graphs are horizontal lines, and the limit set does not depend on Λ , i.e. the family of representations is constant.

Step 3. Improving the equivariant map to a holomorphic motion of ν_λ (Double ergodicity, a reflected random walk, and the persistence of isolated intersections are used to rule out collisions between the holomorphic graphs).

More precisely here we show that there exists a discrete subset $F \subset U$, such that, outside F , the support of the stationary measure moves holomorphically. Furthermore, this holomorphic motion is G -invariant. With θ as in Step 2, we define $\theta_r \subset U \times \mathbb{P}^1$ to be the graph of $\theta(r, \cdot)$.

The *reflected measure* $\check{\mu}$ is the push-forward of μ under $g \mapsto g^{-1}$. The associated Lyapunov exponent $\check{\chi}(\lambda)$ actually equals χ since for $g \in \text{PSL}(2, \mathbb{C})$, $\|g\| = \|g^{-1}\|$. In particular, $\check{T}_{\text{bif}} = 0$

in U . Finally, we can define a map $\check{\theta} : P(G, \check{\mu}) \times U \rightarrow \mathbb{P}^1$ as in lemma 3.10 and the equivariant family of graphs $\check{\theta}_{\check{r}}$ associated to it.

On the product $P(G, \mu) \times P(G, \check{\mu})$ we fix the measure class of the product of the Markov measures starting from e on the corresponding Poisson boundaries. Let $D \Subset U$ be a set, and $\iota_D : P(G, \mu) \times P(G, \check{\mu}) \rightarrow \mathbb{N}$ the measurable map defined almost everywhere by letting $\iota_D(r, \check{r})$ be the number of isolated intersection points (with multiplicity) of the graphs θ_r and $\check{\theta}_{\check{r}}$ in $\pi_1^{-1}(D)$. This number is finite since $D \Subset U$, and is invariant under the diagonal action of G , i.e. $\iota_D(r, \check{r}) = \iota_D(gr, g\check{r})$, by equivariance of the maps θ and $\check{\theta}$. By the double ergodicity theorem of Kaimanovich [Kai], ι_D is a.e. equal to a constant, which will simply be denoted by ι_D .

We see that $D \mapsto \iota_D$ defines an integer-valued measure, which is finite on relatively compact subsets. It is then straightforward to show that it must be a sum of Dirac masses with integer coefficients, supported on a discrete set F .

For the reader's convenience, let us recall the idea of the proof of double ergodicity. To a given bi-infinite sequence $h = (h_n)_{n \in \mathbb{Z}}$, we associate two sequences $h^+ = (h_n)_{n \geq 1}$ and $h^- = (h_n^{-1})_{n \geq 0}$, hence two points $r = r(h)$ and $\check{r} = r(\check{h})$ in the respective Poisson boundaries $P(G, \mu)$ and $P(G, \check{\mu})$. The map $h \mapsto (r, \check{r})$ sends the measure $\mu^{\mathbb{Z}}$ on $G^{\mathbb{Z}}$ to a measure in the measure class of $P(G, \mu) \times P(G, \check{\mu})$. Now, if $\sigma(h_n) = (h_{n+1})$ is the bilateral shift acting on $G^{\mathbb{Z}}$, we have the immediate formulas $r(\sigma h) = h_1^{-1}r(h)$ and $\check{r}(\sigma h) = h_1^{-1}\check{r}(h)$. Hence we deduce that the function $\iota_D(r, \check{r})$ on $G^{\mathbb{Z}}$ is invariant under the bilateral shift, hence constant $\mu^{\mathbb{Z}}$ -a.e. by ergodicity. We conclude that ι_D is almost everywhere constant on $P(G, \mu) \times P(G, \check{\mu})$.

Fix an open subset D disjoint from F . Reducing D if necessary, we can find three disjoint graphs in the family $\{\theta_r, r \in \Omega'\}$. Indeed, if $\lambda_q \in D$ is a parameter from the dense sequence considered in the proof of Lemma 3.10, we know that $(\theta(\cdot, \lambda_q))_* \mathbf{P}_e$ is the stationary measure ν_{λ_q} . This measure is diffuse so this gives us three parameters r_i for which the points $\theta(r_i, \lambda_q)$ are disjoint. If D is small enough, the associated graphs will be disjoint as well.

If the r_i are chosen generically, there exists a set $\check{\Omega}$ of full measure such that for $\check{r} \in \check{\Omega}$, $\check{\theta}_{\check{r}}$ avoids these three disjoint graphs. We conclude that the $\check{\theta}_{\check{r}}$, for $\check{r} \in \check{\Omega}$, form a normal family. Reversing the argument, we obtain a set Ω of full measure such that the associated θ_r also form a normal family.

A consequence of this is that for each $\lambda \in D$, $(\theta(\cdot, \lambda))_* \mathbf{P}_e = \nu_{\lambda}$. Indeed, we know that this equality is true on a dense subset. Furthermore, the right hand side is continuous in λ by uniqueness of the stationary measure, and so is the left hand side by the normality of the family of graphs. Likewise, $(\check{\theta}(\cdot, \lambda))_* \check{\mathbf{P}}_e = \check{\nu}_{\lambda}$.

Let Θ denote the family of graphs $\{\theta_r, r \in \Omega\}$ (and similarly, $\check{\Theta}$ for $\{\check{\theta}_{\check{r}}\}$). At this point we know that there exist full measure subsets $\Omega \subset G^{\mathbb{N}}$ (resp. $\check{\Omega} \subset G^{\mathbb{N}}$) such that for a.e. $(r, \check{r}) \in \Omega \times \check{\Omega}$, θ_r and $\check{\theta}_{\check{r}}$ do not intersect in $\pi_1^{-1}(D)$. Notice that if μ is symmetric (i.e. $\mu = \check{\mu}$) at this point we can simply take the closure to obtain the desired holomorphic motion. The general case requires a few more arguments.

We claim that if $\theta \in \overline{\Theta}$, the set of r 's such that θ_r is different from θ_0 and contained in a given tubular neighborhood of θ_0 has positive measure. To see this, fix a large constant C , larger than the volume of θ_0 , and restrict the attention to the set $\Theta_C \subset \Theta$ of graphs whose volume is not greater than C . By Step 2, $\mu^n(\Theta_C) \geq 1 - \varepsilon$ when C is large. The space of graphs of volume $\leq C$, equipped with the convergence on compact subsets of D is a compact metrizable space. Pushing \mathbf{P}_e under θ gives rise to a measure on this space, and our claim

comes down to saying that the support of this measure has no isolated points. For this, observe that more generally $\check{\theta}_* \check{P}_e$ has no atoms, for otherwise since $(\check{\theta}(\cdot, \lambda))_* \check{P}_e = \check{\nu}_\lambda$, such an atom would give rise to an atom of ν_λ , which does not happen by Furstenberg's Theorem 2.8.

Let $(\theta, \check{\theta}) \in \overline{\Theta} \times \overline{\check{\Theta}}$. If θ and $\check{\theta}$ admit an isolated intersection, then by the continuity of the intersection number of analytic subvarieties [Ch, §12.3, Corollary 4], the same is true for any pair of graphs $(\theta', \check{\theta}')$ close to $(\theta, \check{\theta})$ in the Hausdorff topology. By the previous observation, we obtain a set of positive $P_e \otimes \check{P}_e$ measure of intersecting pairs, which is contradictory. We conclude that any two such θ and $\check{\theta}$ are either disjoint or equal.

Now fix $\lambda_0 \in D$ and $z_0 \in \text{Supp}(\nu_{\lambda_0})$. Since Θ is a normal family, there exists $\theta \in \overline{\Theta}$, passing through (λ_0, z_0) . But since the measures ν_{λ_0} and $\check{\nu}_{\lambda_0}$ have the same support there also exists $\check{\theta} \in \overline{\check{\Theta}}$ through (λ_0, z_0) . Thus, by the previous paragraph, $\theta = \check{\theta}$. We conclude that $\text{Supp}(\nu_\lambda)$ moves holomorphically over D . The invariance of this holomorphic motion follows from the equivariance of θ .

Step 4. Concluding stability from the motion of $\text{Supp}(\nu_\lambda)$.

Let as above $D \subset \Lambda$ be a domain disjoint from the discrete exceptional set F . Being a closed invariant set, $\text{Supp}(\nu_\lambda)$ contains all fixed points of loxodromic and parabolic elements. For $\lambda_0 \in D$, let $q(\lambda_0)$ be a fixed point, say attracting, of a loxodromic element $\rho_{\lambda_0}(g)$. It admits a natural holomorphic continuation $q(\lambda)$ as a fixed point in a neighborhood of λ_0 in D . Let also γ be the graph of the holomorphic motion of $\text{Supp}(\nu_\lambda)$ through $q(\lambda_0)$, constructed in step 3. With notation as before, by invariance of the holomorphic motion, we have that $\widehat{g}(\gamma) = \gamma$. On the other hand, near λ_0 , since $q(\lambda)$ stays attracting, $\widehat{g}^n(\gamma)$ converges to q . Hence $\gamma \equiv q$ near λ_0 . By analytic continuation we thus infer that $\gamma(\lambda)$ is a fixed point of $\rho_\lambda(g)$ throughout D .

Reversing the argument shows that for all $\lambda \in D$, $\rho_\lambda(g)$ stays loxodromic. Indeed the above reasoning first implies that the two fixed points of $\rho_\lambda(g)$ remain distinct throughout D . Furthermore, if $p(\lambda_0) \in \mathbb{P}^1$ is any point of $\text{Supp}(\nu_{\lambda_0})$ different from the other fixed point of $\rho_{\lambda_0}(g)$, and $\lambda \mapsto p(\lambda)$ denotes its continuation along the holomorphic motion, then by normality the sequence of graphs $\widehat{g}^n(p)$ converges to $q(\lambda)$ locally uniformly on D . This shows that for $\lambda \in D$, $\rho_\lambda(g)$ is never elliptic.

Since the same reasoning is valid for parabolic transformations, we see that the Möbius transformations $\rho_\lambda(g)$ stay of constant type as λ ranges along D , and we conclude that D is contained in the stability locus. In particular, for $\lambda \in D$, ρ_λ is discrete and faithful.

It remains to show that the exceptional set F is empty. Let $\lambda_0 \in F$. By the Jørgensen (Margulis-Zassenhaus) theorem [Kap, p. 170], ρ_{λ_0} is discrete and faithful, so ρ_λ is discrete and faithful in the neighborhood of λ_0 , and finally $\lambda_0 \in \text{Stab}$. Thus we have shown that $U \subset \text{Stab}$, thereby concluding the proof of the theorem. \square

We now show that Theorem 3.9 remains true for generally non-elementary families, that is, when a proper subset of Λ is made of elementary representations. This is the case for instance for the universal family of representations of G into $\text{PSL}(2, \mathbb{C})$ (possibly after desingularization).

Theorem 3.13. *Let (G, μ, ρ) be a holomorphic family of representations, which is non-trivial, generally faithful and generally non-elementary, endowed with a probability measure μ , generating G as a semi-group, and satisfying (8).*

Then the support of T_{bif} coincides with the bifurcation locus.

Proof. Assume that the subset $E \subset \Lambda$ of elementary representations is non-empty, and different from Λ . In particular E is contained in the bifurcation locus. We need to show that $E \subset \text{Supp}(T_{\text{bif}})$.

Using the terminology introduced in the proof of Lemma 2.4, E decomposes as $E = E_{\text{I}} \cup E_{\text{II}}$ where E_{I} (resp. E_{II}) is a proper analytic (resp. real analytic) subset of type I (resp. type II) points. It is easy to see that χ extends continuously by 0 on E_{II} . In particular that χ cannot be harmonic near E_{II} , so $E_{\text{II}} \subset \text{Supp}(T_{\text{bif}})$.

Assume now that $\lambda_0 \in E_{\text{I}} \setminus E_{\text{II}}$. Fix a neighborhood $N \ni \lambda_0$ such that $N \cap E_{\text{II}} = \emptyset$. We claim that in N , $E \subset \text{Bif}|_{\Lambda \setminus E}$. This clearly implies that $E \cap N \subset \text{Supp}(T_{\text{bif}})$. To prove the claim, notice that since $E \subsetneq \Lambda$, there exists $g, h \in G$ such that $\lambda \mapsto \text{tr}^2[g_\lambda, h_\lambda]$ is not constant so there are parameters λ close to E where $\text{tr}^2[g_\lambda, h_\lambda] = 4 \cos^2 \theta$, with $\theta/\pi \notin \mathbb{Q}$. By assumption these parameters do not belong to E_{I} so they correspond to non-elementary representations, which are not discrete because they contain an elliptic element of infinite order, and we are done. \square

3.4. Classification of stationary currents on $\Lambda \times \mathbb{P}^1$. In view of the previous results, it is natural to wonder whether it is possible for a holomorphic family of representations to admit a *stationary current*, that is a positive closed (1,1) current \hat{T} on $\Lambda \times \mathbb{P}^1$ such that (with notation as before) $\int \hat{g}_* \hat{T} d\mu(g) = \hat{T}$.

Let us keep hypotheses as in §3.3. We say that a current on $\Lambda \times \mathbb{P}^1$ is vertical if it is an integral of currents of integration over vertical fibers, that is, a current of the form $\pi_1^* T$, with T a closed positive current on Λ . Equivalently (see Lemma 3.7), \hat{T} is vertical if $\hat{T} \wedge \pi_1^* \omega^k = 0$ ($k = \dim(\Lambda)$). Every vertical current is stationary.

Another possibility for the existence of a stationary current is when the family of representations is stable over Λ . Then it is clear that the family of stationary measures ν_λ is invariant under the holomorphic motion conjugating the representations. Fix a parameter $\lambda_0 \in \Lambda$, and for $z \in \mathbb{P}^1$, let $\Gamma_z \subset \Lambda \times \mathbb{P}^1$ be the graph of the holomorphic motion passing through (λ_0, z) . Consequently, we can define a stationary current by setting $\hat{T} = \int [\Gamma_z] d\nu_{\lambda_0}(z)$.

The following result says that essentially all stationary currents are of this form.

Theorem 3.14. *Let (G, μ, ρ) be an admissible family of representations, satisfying the exponential moment assumption (8). Assume further that the stability locus is not empty.*

Assume that there exists a stationary current \hat{T} in $\Lambda \times \mathbb{P}^1$. Then either \hat{T} is vertical or the family is stable over Λ and \hat{T} is the current made of the family of holomorphically varying stationary measures as above.

We believe that the additional assumption that Stab is non-empty is unnecessary.

Another interpretation of this result is the following. We say that a family of measures $\{m_\lambda\}_{\lambda \in \Lambda}$ on $\{\lambda\} \times \mathbb{P}^1$ *varies holomorphically* if the m_λ are vertical slices of a positive closed current in $\Lambda \times \mathbb{P}^1$ (see the discussion on structural varieties in the space of positive measures on \mathbb{P}^1 in [DS1, §A.4]). What the theorem says is that the natural holomorphic family of stationary measures over the stability locus can never be holomorphically continued across the boundary of the stability locus.

Proof. Let us first recall some classical facts on currents on $\Lambda \times \mathbb{P}^1$. If m is any probability measure with compact support in Λ , then the mass of $\hat{T} \wedge \pi_1^* m$ is a constant independent of m , called the *slice mass* of \hat{T} . Indeed if m_1 and m_2 are smooth probability measures (viewed as $2k$ -forms on Λ), then $m_1 - m_2 = d\theta$, where θ is a compactly supported $(2k-1)$ -form, and

$\langle \widehat{T}, d\theta \rangle = 0$. For Lebesgue a.e. $\lambda \in \Lambda$, the slice measure $\widehat{T} \wedge [\{\lambda\} \times \mathbb{P}^1]$ is well-defined, and by the above discussion, its mass does not depend on λ . In particular if \widehat{T} is not vertical the slice mass is non-zero and we may assume that it equals 1.

We assume that the family admits a non-vertical stationary current, and will show that it is stable. Assume first that $\dim(\Lambda) = 1$. Since \widehat{T} is stationary, for a.e. λ , $\widehat{T} \wedge [\{\lambda\} \times \mathbb{P}^1]$ must be the unique stationary probability measure with respect to the action of $\rho_\lambda(G)$. For every $n \geq 1$ we have that $\int (g_1 \cdots g_n)_* \widehat{T} d\mu(g_1) \cdots d\mu(g_n) = \widehat{T}$. Therefore, arguing as in Step 2 of the proof of Theorem 3.9, there exists a set $\Omega \subset G^\mathbb{N}$ of full measure such that if $\mathbf{g} \in \Omega$, there exists a subsequence n_j such that the sequence of currents $(\widehat{r}_{n_j})_* \widehat{T}$ has bounded mass (recall that $r_n = r_n(\mathbf{g}) = g_1 \cdots g_n$). Let U be an open set contained in the stability locus, and $\lambda_0 \in U$. As before, if $z \in \mathbb{P}^1$ let Γ_z be the graph over U , passing through (λ_0, z) , subordinate to the holomorphic motion conjugating the representations.

Working in $\{\lambda_0\} \times \mathbb{P}^1$, we know that for a.e. $\mathbf{g} \in G^\mathbb{N}$, $(g_{\lambda_0,1})_* \cdots (g_{\lambda_0,n})_* \nu_{\lambda_0}$ converges to a Dirac mass $\delta_{z(\mathbf{g}, \lambda_0)}$ of law ν_{λ_0} . Since the representations are conjugate over U , we conclude that there exists a set $\Omega' \subset G^\mathbb{N}$ of full measure such that if $\mathbf{g} \in \Omega'$ $(\widehat{r}_n)_* \widehat{T}$ converges to $[\Gamma_{z(\mathbf{g}, \lambda_0)}] =: [\Gamma_{\mathbf{g}}]$ in $U \times \mathbb{P}^1$.

Putting the two previous paragraphs together (and extracting again if necessary), we see that if $\mathbf{g} \in \Omega \cap \Omega'$, there exists a subsequence n_j such that $(\widehat{r}_{n_j})_* \widehat{T}$ converges to some \widehat{S} , with $\widehat{S} = [\Gamma_{\mathbf{g}}]$ in $U \times \mathbb{P}^1$.

We claim that $[\Gamma_{\mathbf{g}}]$ admits a continuation as a graph over Λ . Indeed, by Siu's Decomposition Theorem [Si], $\widehat{S} = S_1 + S_2$, where S_1 is a current of integration over an at most countable family of analytic subsets, and S_2 gives no mass to curves. Thus, there exists an irreducible analytic subset V of $\Lambda \times \mathbb{P}^1$, continuing $\Gamma_{\mathbf{g}}$. Notice that V is a branched cover over Λ relative to π_1 . Since $\widehat{S} \geq [V]$, we see that $V = \Gamma_{\mathbf{g}}$ in $U \times \mathbb{P}^1$, hence V must be graph over Λ , and we are done.

In this way we construct a family of graphs, parameterized by a full measure subset of $G^\mathbb{N}$, which is equivariant since it is equivariant over U . So we are exactly in the same situation as in Step 3 of the proof of Theorem 3.9, and we conclude that the family of representations is stable over Λ . This settles the case where $\dim(\Lambda) = 1$.

To handle the general case we use a slicing argument (see [DS1, §A.3] for basics on slicing closed positive currents). It is no loss of generality to assume that Λ is an open ball in \mathbb{C}^k . Assume as before that the family admits a non-vertical stationary current, and suppose by contradiction that the bifurcation locus is non-empty. Then by the Margulis Zassenhaus lemma, there exists an open subset V in Λ that is disjoint from the set of discrete and faithful representations. Now consider a linear projection $p : \Lambda \rightarrow \mathbb{C}^{k-1}$, having the property that an open set of fibers intersects both V and the stability locus, and define $\widehat{p} : \Lambda \times \mathbb{P}^1 \rightarrow \mathbb{C}^{k-1}$ by $\widehat{p} = p \circ \pi$.

For (Lebesgue) a.e. $x \in \mathbb{C}^{k-1}$ the slice $\widehat{T}|_{\widehat{p}^{-1}(x)}$ of \widehat{T} along the fiber $\widehat{p}^{-1}(x) = p^{-1}(x) \times \mathbb{P}^1$ is a well defined closed positive current, which is a.s. stationary since the group action preserves the fibers. The proof will be finished if we can show that for a.e. x , $\widehat{T}|_{\widehat{p}^{-1}(x)}$ is not vertical. Indeed, we would then have a set of positive measure of fibers $p^{-1}(x)$ intersecting both V and the stability locus, and for which there exists a non-vertical stationary current on $p^{-1}(x) \times \mathbb{P}^1$, thereby contradicting the previously treated case $\dim(\Lambda) = 1$.

To show that $\widehat{T}|_{\widehat{p}^{-1}(x)}$ is not vertical, we show that it has positive slice mass (relative to the projection $\pi : \widehat{p}^{-1}(x) \rightarrow p^{-1}(x)$). Recall that \widehat{T} is supposed to have slice mass 1. The so-called *slicing formula* asserts that if Ω is a positive test form of maximal degree on \mathbb{C}^{k-1} of total mass 1 (which can be identified to a probability measure), and ϕ is any test form of bidegree $(1,1)$ on $\Lambda \times \mathbb{P}^1$, we have

$$\int \left(\int_{\widehat{p}^{-1}(x)} \widehat{T}|_{\widehat{p}^{-1}(x)} \wedge \phi \right) \Omega(x) = \int \widehat{T} \wedge \phi \wedge (\widehat{p})^* \Omega.$$

Now if locally we view Λ as a product $\mathbb{C}^{k-1} \times \mathbb{C}$ with respective first and second projections p and q (thus identifying under q the fibers $p^{-1}(x)$ with the second factor), and if we specialize the above formula to forms ϕ of the form $\pi^* q^* \varphi$ with φ a positive test $(1,1)$ form of total mass 1 on \mathbb{C} , we get that for a.e. x , and every such φ , $\int_{\widehat{p}^{-1}(x)} \widehat{T}|_{\widehat{p}^{-1}(x)} \wedge \pi^* q^* \varphi = 1$, which was the desired result. \square

4. EQUIDISTRIBUTION THEOREMS

In this section we prove several equidistribution results in parameter space, including Theorems B and C. We also give another geometric description of T_{bif} , in the spirit of Theorem 3.5, where the approximating varieties are now fixed points of fibered Möbius transformations.

4.1. A general equidistribution scheme. The following theorem may be interpreted as a general method for proving equidistribution results associated to random sequences in parameter space. Specializing it to well chosen functions F will lead to various equidistribution statements, including Theorem B.

Theorem 4.1. *Let (G, μ, ρ) be an admissible families of representations of G . Let F be a psh function on $\text{PSL}(2, \mathbb{C})^k$. Assume that:*

- i. There exist non negative real numbers a_1, \dots, a_k , with $\sum a_i = 1$, and a constant C such that for $(\gamma_i)_{i=1}^k \in \text{PSL}(2, \mathbb{C})^k$,*

$$F(\gamma_1, \dots, \gamma_k) \leq a_1 \log \|\gamma_1\| + \dots + a_k \log \|\gamma_k\| + C.$$

- ii. If $\lambda \in \Lambda$ is fixed, then for $(\mu^{\mathbb{N}})^{\otimes k}$ -a.e. $(\mathbf{g}_1, \dots, \mathbf{g}_k)$,*

$$\frac{1}{n} F(\rho_\lambda(l_n(\mathbf{g}_1)), \dots, \rho_\lambda(l_n(\mathbf{g}_k))) \xrightarrow{n \rightarrow \infty} \chi(\lambda).$$

Then for $(\mu^{\mathbb{N}})^{\otimes k}$ -a.e. $(\mathbf{g}_1, \dots, \mathbf{g}_k)$, the sequence of psh functions defined by

$$\lambda \mapsto \frac{1}{n} F(\rho_\lambda(l_n(\mathbf{g}_1)), \dots, \rho_\lambda(l_n(\mathbf{g}_k)))$$

converges to $\chi(\lambda)$ in $L_{\text{loc}}^1(\Lambda)$.

One might also specify different measures μ_i on each factor. In this case the $\chi(\lambda)$ in *ii.* must be replaced by $\sum a_i \chi(\lambda, \mu_i)$ and the same function will appear in the conclusion.

The starting point is the following proposition. Recall from Theorem 2.8 that for a fixed representation, for $\mu^{\mathbb{N}}$ -a.e. \mathbf{g} , $\frac{1}{n} \log \|\rho(l_n(\mathbf{g}))\|$ converges to $\chi(\rho)$. We now give a parameterized version of this result.

Proposition 4.2. *Let (G, μ, ρ) be an admissible family of representations of G into $\text{PSL}(2, \mathbb{C})$. Then for $\mu^{\mathbb{N}}$ -a.e. $\mathbf{g} \in G^{\mathbb{N}}$, the sequence of functions $\lambda \mapsto \frac{1}{n} \log \|\rho_\lambda(l_n(\mathbf{g}))\|$ converges to $\chi(\lambda)$ in $L_{\text{loc}}^1(\Lambda)$.*

Proof. Of course, the point is to make a choice of generic \mathbf{g} not depending on λ . It is no loss of generality to assume that Λ is a ball in $\mathbb{C}^{\dim(\Lambda)}$. Let $U \subset \Lambda$ be any open subset, and for $\mathbf{g} \in G^{\mathbb{N}}$ consider the sequence $\Theta_n(U)$ defined by $\Theta_n(\mathbf{g}, U) = \int_U \log \|\rho_\lambda(l_n(\mathbf{g}))\| d\lambda$. By (6), this assignment defines a real valued sub-additive cocycle, which, by the moment condition (2), satisfies $\int \Theta_1(\mathbf{g}, U) d\mu^{\mathbb{N}}(\mathbf{g}) < \infty$. Therefore, by Kingman's sub-additive ergodic theorem and the ergodicity of the shift acting on $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$, we deduce that $\frac{1}{n}\Theta_n(\mathbf{g}, U)$ converges $\mu^{\mathbb{N}}$ -a.e. to a non-negative number $\Theta(U)$ independent of \mathbf{g} .

Also, $\frac{1}{n} \int \Theta_n(\mathbf{g}, U) d\mu^{\mathbb{N}}(\mathbf{g})$ converges to $\Theta(U)$. Indeed $0 \leq \Theta_n(\mathbf{g}, U) \leq C \text{length}(l_n \mathbf{g})$ for some constant C , whereas by Kingman's theorem and (2) the sequence $\text{length}(l_n(\mathbf{g}))$ converges in $L^1(\mu^{\mathbb{N}})$ (see the domination argument after Theorem 2.8). Therefore, the convergence of $\frac{1}{n}\Theta_n(\mathbf{g}, U)$ to $\Theta(U)$ takes place in $L^1(\mu^{\mathbb{N}})$.

Take now a countable neighborhood basis $(U_q)_q$ of Λ . There exists a full measure subset $\Omega \subset G^{\mathbb{N}}$ such that if $\mathbf{g} \in \Omega$, then for every q , $\frac{1}{n} \int_{U_q} \log \|\rho_\lambda(l_n(\mathbf{g}))\| d\lambda$ converges to some $\Theta(U_q)$.

By (2) again, for $\mu^{\mathbb{N}}$ -a.e. \mathbf{g} the length of the word $l_n(\mathbf{g})$ in G grows at linear speed. Hence the sequence of psh functions on Λ defined by $(\lambda \mapsto \frac{1}{n} \log \|\rho_\lambda(l_n(\mathbf{g}))\|)_n$ is locally uniformly bounded, so it admits convergent subsequences in L^1_{loc} . If $\theta(\mathbf{g}, \lambda)$ denotes any of its cluster values, we see that $\int_{U_q} \theta(\mathbf{g}, \lambda) d\lambda$ must be equal to $\Theta(U_q)$. Hence the sequence actually converges to a limit independent of \mathbf{g} , which we denote by $\theta(\lambda)$.

The last step is of course to prove that $\theta(\cdot) = \chi(\cdot)$. For this, it is enough to integrate with respect to \mathbf{g} . Indeed, for any $\lambda \in \Lambda$,

$$\frac{1}{n} \int_G \log \|\rho_\lambda(g)\| d\mu^n(g) = \frac{1}{n} \int \log \|\rho_\lambda(l_n(\mathbf{g}))\| d\mu^{\mathbb{N}}(\mathbf{g}) \xrightarrow{n \rightarrow \infty} \chi(\lambda).$$

Since the left hand side is locally uniformly bounded in n , by dominated convergence we infer that for any open set U ,

$$\int_U \left(\frac{1}{n} \int \log \|\rho_\lambda(l_n(\mathbf{g}))\| d\mu^{\mathbb{N}}(\mathbf{g}) \right) d\lambda \xrightarrow{n \rightarrow \infty} \int_U \chi.$$

Now we let $U = U_q$ and switch the integrals to see that

$$\int_U \left(\frac{1}{n} \int \log \|\rho_\lambda(l_n(\mathbf{g}))\| d\mu^{\mathbb{N}}(\mathbf{g}) \right) d\lambda = \int \frac{1}{n} \Theta_n(\mathbf{g}, U_q) d\mu^{\mathbb{N}}(\mathbf{g}),$$

which converges to $\Theta(U_q) = \int_{U_q} \theta$. We conclude that for any q , $\int_{U_q} \theta = \int_{U_q} \chi$, and the result follows. \square

We also need the following variation on the Hartogs Lemma (see [H2, pp. 149-151]).

Lemma 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a connected open set and (v_n) be a sequence of psh functions in Ω converging in L^1_{loc} to a continuous psh function v . Assume now that (u_n) is another sequence of psh functions in Ω such that:*

- for every $x \in \Omega$, $u_n(x) \leq v_n(x)$;
- there exists a dense subset $D \subset \Omega$ such that for every $x \in D$, $u_n(x) \rightarrow v(x)$ as $n \rightarrow \infty$.

Then (u_n) converges to v in L^1_{loc} .

Proof. Observe first that (v_n) is locally uniformly bounded above, hence so is (u_n) . Since $u_n(x) \rightarrow v(x)$ on D , (u_n) cannot diverge to $-\infty$ so there exists a subsequence (u_{n_j}) converging in L^1_{loc} to a psh function u . The point is to prove that $u = v$.

For a.e. x , $\limsup u_{n_j} = u(x)$, from which we infer that $u \leq v$ a.e. Now suppose that there exists x_0 such that $u(x_0) < v(x_0)$. By upper semi-continuity (we use the fact that v is continuous) there exists a relatively compact open set $B \ni x_0$ where $u < v - \delta$ for some positive δ . By the Hartogs lemma for large j we get that $u_{n_j} < v - \delta$ on B . This contradicts the fact that $u_n(x) \rightarrow v(x)$ on a dense subset. \square

Proof of Theorem 4.1. Pick a dense sequence (λ_p) in Λ . There exists a set $\Omega_0 \subset (G^\mathbb{N})^k$ of full measure such that if $(\mathbf{g}_1, \dots, \mathbf{g}_k) \in \Omega_0$, then for every p

$$\frac{1}{n} F(\rho_{\lambda_p}(l_n(\mathbf{g}_1)), \dots, \rho_{\lambda_p}(l_n(\mathbf{g}_k))) \xrightarrow{n \rightarrow \infty} \chi(\lambda_p).$$

Applying Proposition 4.2, let $\Omega_1^k \subset G^\mathbb{N}$ be a set of full measure such that for any $(\mathbf{g}_1, \dots, \mathbf{g}_k) \in \Omega_1$,

$$\frac{a_1}{n} \log \|\rho_\lambda(l_n(\mathbf{g}_1))\| + \dots + \frac{a_k}{n} \log \|\rho_\lambda(l_n(\mathbf{g}_k))\| \rightarrow \chi(\lambda) \text{ in } L_{\text{loc}}^1.$$

Now for every $\lambda \in \Lambda$ we have that

$$\frac{1}{n} F(\rho_\lambda(l_n(\mathbf{g}_1)), \dots, \rho_\lambda(l_n(\mathbf{g}_k))) \leq \frac{a_1}{n} \log \|\rho_\lambda(l_n(\mathbf{g}_1))\| + \dots + \frac{a_k}{n} \log \|\rho_\lambda(l_n(\mathbf{g}_k))\| + O\left(\frac{1}{n}\right).$$

From Lemma 4.3, we thus conclude that if $(\mathbf{g}_1, \dots, \mathbf{g}_k) \in \Omega_0 \cap \Omega_1^k$,

$$\frac{1}{n} F(\rho_\lambda(l_n(\mathbf{g}_1)), \dots, \rho_\lambda(l_n(\mathbf{g}_k))) \rightarrow \chi(\lambda) \text{ in } L_{\text{loc}}^1,$$

which finishes the proof. \square

As a sample application of Theorem 4.1, let us prove the following variant of Theorem 3.5.

Theorem 4.4. *Let (G, μ, ρ) be an admissible family of representations of G into $\text{PSL}(2, \mathbb{C})$. Fix $z_0 \in \mathbb{P}^1$. Then for $\mu^\mathbb{N}$ -a.e. $\mathbf{g} \in G^\mathbb{N}$, the sequence of currents $\frac{1}{n} [\widehat{l_n(\mathbf{g})}] \cdot z_0$ in $\Lambda \times \mathbb{P}^1$ converges to $\pi_1^*(T_{\text{bif}})$.*

By following step by step the proof of Theorem 3.5 (and keeping notation as in that proof), we first see that every cluster value \widehat{S} of the sequence $\frac{1}{n} [\widehat{l_n(\mathbf{g})}] \cdot z_0$ must satisfy $\langle \widehat{S}, \pi_1^* \omega^k \rangle = 0$. To show that this sequence of currents is a.s. of bounded mass and converges to $\pi_1^*(T_{\text{bif}})$, it is enough to show that for every $(k-1, k-1)$ test form ϕ on Λ and a.e. \mathbf{g}

$$\left\langle (\pi_1)_* \left(\frac{1}{n} [\widehat{l_n(\mathbf{g})}] \cdot z_0 \right) \wedge \pi_2^* \omega_{\mathbb{P}^1}, \phi \right\rangle = \int_\Lambda \phi \wedge dd_\lambda^c \left(\frac{1}{n} \log \frac{\|\rho_\lambda(l_n(\mathbf{g})) Z_0\|}{\|Z_0\|} \right) \xrightarrow{n \rightarrow \infty} \langle T_{\text{bif}}, \phi \rangle,$$

where the equality on the left hand side is obtained as in (16). Thus we conclude that to obtain Theorem 4.4 it is enough to establish the following:

Proposition 4.5. *Let (G, μ, ρ) be an admissible family of representations of G into $\text{PSL}(2, \mathbb{C})$. Let $z_0 \in \mathbb{P}^1$ and let $Z_0 \in \mathbb{C}^2$ be any lift of z_0 . Then for $\mu^\mathbb{N}$ a.e. \mathbf{g} the sequence of functions $\lambda \mapsto \frac{1}{n} \log \|\rho_\lambda(l_n(\mathbf{g})) Z_0\|$ converges to $\chi(\lambda)$ in $L_{\text{loc}}^1(\Lambda)$.*

Proof. It is enough to check that the assumptions of Theorem 4.1 hold for the psh function $F : \gamma \mapsto \log \|\gamma Z_0\|$ on $\text{PSL}(2, \mathbb{C})$. The inequality in *i.* is obvious, while *ii.* follows from Theorem 2.8. The result follows. \square

The next result shows that under an additional assumption we can integrate with respect to \mathbf{g} in Theorem 4.1. It is slightly more convenient to state it in terms of currents rather than potentials. We use the notation $\mathbf{M}_\Omega(T)$ for the mass of the current T in Ω .

Proposition 4.6. *Let (G, μ, ρ) be an admissible family of representations, and F a function on $\mathrm{PSL}(2, \mathbb{C})^k$ satisfying the assumptions of Theorem 4.1. For $(g_1, \dots, g_k) \in G^k$ let $T(g_1, \dots, g_k)$ be the current on Λ defined by $T(g_1, \dots, g_k) = dd_\lambda^c(F(\rho_\lambda(g_1), \dots, \rho_\lambda(g_k)))$.*

Assume that for every $(g_1, \dots, g_k) \in G^k$ and every $\Lambda' \Subset \Lambda$, there exists a constant $C(\Lambda')$ such that $\mathbf{M}_{\Lambda'}(T(g_1, \dots, g_k)) \leq C \sum_{i=1}^k \text{length}(g_i)$.

Then $\frac{1}{n}T(l_n(\mathbf{g}_1), \dots, l_n(\mathbf{g}_k))$ converges to T_{bif} in $L^1(\mu^{\mathbb{N}} \otimes \dots \otimes \mu^{\mathbb{N}})$. In particular

$$(21) \quad \frac{1}{n} \int T(g_1, \dots, g_k) d\mu^n(g_1) \cdots d\mu^n(g_k) \xrightarrow{n \rightarrow \infty} T_{\text{bif}}.$$

Proof. Note first that (21) means that for any $(k-1, k-1)$ test form φ on Λ ($k = \dim(\Lambda)$),

$$(22) \quad \frac{1}{n} \int \langle T(g_1, \dots, g_k), \varphi \rangle d\mu^n(g_1) \cdots d\mu^n(g_k) \xrightarrow{n \rightarrow \infty} \langle T_{\text{bif}}, \varphi \rangle.$$

The mass estimate in the statement of the proposition implies that $|\langle T(g_1, \dots, g_k), \varphi \rangle| \leq C(\varphi) \sum_{i=1}^k \text{length}(g_i)$. Since an admissible family of representations satisfies (2), this guarantees the existence of the integrals in (22). Next, by Theorem 4.1, for a.e. $(\mathbf{g}_1, \dots, \mathbf{g}_k)$, $\frac{1}{n} \langle T(l_n(\mathbf{g}_1), \dots, l_n(\mathbf{g}_k)), \varphi \rangle$ converges to $\langle T_{\text{bif}}, \varphi \rangle$. To get the desired result we need to show that this convergence takes place in $L^1(\mu^{\mathbb{N}} \otimes \dots \otimes \mu^{\mathbb{N}})$. Now the domination argument after Theorem 2.8 implies that $\frac{1}{n} \text{length}(l_n(\mathbf{g}))$ converges in $L^1(\mu^{\mathbb{N}})$ to a constant, so the result simply follows from the Dominated Convergence Theorem. \square

4.2. Equidistribution of parameters with a given trace. Let (G, μ, ρ) be an admissible family of representations, and fix $t \in \mathbb{C}$. Let $\mathcal{P}_t \subset G$ be the set of elements g such that the function $\lambda \mapsto \text{tr}^2(g_\lambda)$ is constant and equal to t (typically, a persistently parabolic element). By Corollary A.2, $\mu^n(\mathcal{P}_t)$ converges to zero.

Our purpose here is to study the distribution of parameters λ such that there exists $g \in G \setminus \mathcal{P}_t$ with $\text{tr}^2(g_\lambda) = t$. This is mostly interesting when $t = 4 \cos^2\left(2\pi \frac{p}{q}\right)$, since the representations for these parameters exhibit “accidental” new relations (also accidental parabolics when $t = 4$). Recall from Corollary 2.7 that such parameters are dense in the bifurcation locus.

For $g \in G \setminus \mathcal{P}_t$ we let $Z(g, t)$ be the codimension 1 subvariety of parameter space defined as $Z(g, t) = \{\lambda, \text{tr}^2(g_\lambda) - t = 0\}$ (with the corresponding multiplicity, if any). Recall that with our conventions, if $g \in \mathcal{P}_t$, $[Z(g, t)] = 0$.

The next result belongs to the general scheme presented in the previous paragraph.

Theorem 4.7. *Let (G, μ, ρ) be an admissible family of representations satisfying the exponential moment condition (8), and fix $t \in \mathbb{C}$.*

Then for $\mu^{\mathbb{N}}$ -a.e. $\mathbf{g} \in G^{\mathbb{N}}$, the sequence of integration currents $\frac{1}{2n} [Z(l_n(\mathbf{g}), t)]$ converges to T_{bif} .

Proof. We work with potentials so for $g \in G \setminus \mathcal{P}_t$, let $u(\lambda, g, t) = \log |\text{tr}^2(g_\lambda) - t|$ be a psh potential of $Z(g, t)$. Notice that if $g \in \mathcal{P}_t$, $u(\lambda, g, t) \equiv -\infty$, nevertheless this won't affect the argument. Since $dd^c\left(\frac{1}{2n}u(\cdot, l_n(\mathbf{g}), t)\right) = \frac{1}{2n}[Z(l_n(\mathbf{g}), t)]$, to get the desired convergence it suffices to show that for $\mu^{\mathbb{N}}$ -a.e. \mathbf{g} , $\frac{1}{2n}u(\cdot, l_n(\mathbf{g}), t)$ converges to χ in $L^1_{\text{loc}}(\Lambda)$ (in particular $l_n(\mathbf{g}) \notin \mathcal{P}_t$ for large n).

For this, it suffices to apply Theorem 4.1 to the psh function defined on $\mathrm{PSL}(2, \mathbb{C})$ by $F(\gamma) = \frac{1}{2} \log |\text{tr}^2(\gamma) - t|$. Assumption *i.* in that theorem clearly holds, and Theorem 2.10 gives *ii.* \square

It is natural to wonder whether the convergence in the previous theorem can be made more precise. We already observed –see the discussion after Corollary 2.11– that it is not true in general that for a given λ , $\frac{1}{2n} \log |\mathrm{tr}^2(\rho_\lambda(l_n(\mathbf{g})) - t|$ converges to $\chi(\lambda)$ in $L^1(\mu^\mathbb{N})$. Here we show that under some global assumptions on Λ we can indeed integrate with respect to \mathbf{g} in Theorem 4.7.

Theorem 4.8. *Let (G, μ, ρ) be an admissible family of representations satisfying the exponential moment condition (8), and fix $t \in \mathbb{C}$. Suppose in addition that one of the following two conditions is satisfied:*

- i. the family of representations $(\rho_\lambda)_{\lambda \in \Lambda}$ is algebraic;*
- ii. or there exists at least one geometrically finite representation in Λ .*

Then

$$(23) \quad \frac{1}{2n} \int_{G^\mathbb{N}} [Z(l_n(\mathbf{g}), t)] d\mu^\mathbb{N}(\mathbf{g}) = \frac{1}{2n} \int_G [Z(g, t)] d\mu^n(g) \xrightarrow{n \rightarrow \infty} T_{\mathrm{bif}}$$

(recall that if $g \in \mathcal{P}_t$, by definition $[Z(g, t)] = 0$).

The notion of an algebraic family of representations was introduced in §2.2. Observe in particular that condition *i.* is satisfied when Λ is an open subset of the family of all representations of G into $\mathrm{PSL}(2, \mathbb{C})$ (resp. modulo conjugacy). It will also be clear from the proof that *ii.* can be relaxed to only requiring that Λ can be continued to a family containing a geometrically finite representation.

Proof. In view of Proposition 4.6 it is enough to show that for every $g \in G$, and every $\Lambda' \Subset \Lambda$, $\mathbf{M}_{\Lambda'}([Z(g, t)]) \leq C(\Lambda') \mathrm{length}(g)$. Observe that if $g \in \mathcal{P}_t$ this is true by definition.

This is easiest under assumption *i.*, so let us assume that (ρ_λ) is an algebraic family. Recall that $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to $\mathrm{SO}(3, \mathbb{C})$, so that, reducing Λ' if necessary, we view Λ' as an open subset of an affine subvariety in \mathbb{C}^{9k} (k is the number of generators). For $1 \leq i \leq k$, let $(a_{i,j})_{1 \leq j \leq 9}$ be the coefficients corresponding to the generator g_i . If $g \in G$ is any element, it is easy to see that $\mathrm{tr}^2(g) - t$ is a polynomial in the $a_{i,j}$ of degree $O(\mathrm{length}(g))$, so the desired estimate simply follows from Bézout's Theorem and the fact that the volume of an algebraic subvariety is controlled by its degree.

Let us now suppose that *ii.* holds. If $g \in G \setminus \mathcal{P}_t$, let $u(\lambda, g) = \log |\mathrm{tr}^2(g_\lambda) - t|$, be a psh potential of $[Z(g, t)]$. If Λ'' is an open set with $\Lambda' \Subset \Lambda'' \Subset \Lambda$, there exists a constant $C(\Lambda', \Lambda'')$ such that $\mathbf{M}_{\Lambda'}([Z(g, t)]) \leq C \|u(\cdot, g)\|_{L^1(\Lambda'')}$ (see [De, Remark 3.4]), so our task is to control this L^1 norm. Notice further that it is enough to consider the case where $\lambda \mapsto \mathrm{tr}^2(g_\lambda)$ is not constant (for otherwise $[Z(g, t)] = 0$). The following lemma then completes the proof of the theorem. \square

Lemma 4.9. *If there exists a geometrically finite representation in Λ , then for every relatively compact open subset $\Lambda' \subset \Lambda$ there exists a constant C such that for every $g \in G$, if $\lambda \mapsto \mathrm{tr}^2(g_\lambda)$ is not a constant function, then $\|u(\cdot, g)\|_{L^1(\Lambda')} \leq C \mathrm{length}(g)$.*

As the proof will show, it is easy to obtain such an estimate for a family consisting entirely of geometrically finite representations. To handle the general case, we use some classical properties of psh functions, which we remind first.

Lemma 4.10. *Let u be a psh function on a connected complex manifold Ω , and $M > 0$ with $\sup_\Omega u \leq M$. Fix two relatively compact open subsets Ω' and Ω'' of Ω , and let $x_0 \in \Omega''$. Then there exists a constant $A(\Omega', \Omega'')$ such that the following properties hold:*

- i. $\|u\|_{L^1(\Omega')} \leq A \max(|u(x_0)|, M)$;
- ii. $\sup_{\Omega'} u \geq -A \max(|u(x_0)|, M)$.

Proof. This follows from a standard compactness argument: consider the family of psh functions of the form $v = \frac{u}{\max(|u(x_0)|, M)}$. This family is compact in $L^1_{\text{loc}}(\Omega)$, since $\sup_{\Omega} v \leq 1$ and $v(x_0) \geq -1$, whence i. and ii. follows. \square

Proof of Lemma 4.9. First, since $\text{tr}^2(g_{\lambda}) \leq C \|g_{\lambda}\|^2$, there exists a constant $M(\Lambda') > 0$ such that for every $g \in G$,

$$(24) \quad \sup_{\lambda \in \Lambda'} |\text{tr}^2(g_{\lambda}) - t| \leq M^{\text{length}(g)}.$$

Let now λ_0 be a parameter such that ρ_{λ_0} is geometrically finite. Recall that this means that ρ_{λ_0} is discrete, faithful, and that there is a finite sided fundamental domain for the ρ_{λ_0} -action of G on hyperbolic 3-space. In this case it is known (see [R, Theorem 12.7.8]) that given any constant $\ell > 0$ the quotient hyperbolic manifold admits only finitely many closed geodesics of length bounded by ℓ .

Let now Λ'' be a small ball containing λ_0 . Observe that if γ is the closed geodesic in M corresponding to some element g_{λ_0} , then the length of γ is given by $2 \log |\lambda_{\max}(g_{\lambda_0})|$, where as before λ_{\max} denotes an eigenvalue of g_{λ} of maximal modulus. Therefore there is only a finite number of conjugacy classes of elements $g \in G$ such that $|\lambda_{\max}(g_{\lambda_0})| \leq 4 + |t|$ (observe that in a geometrically finite representation there is only a finite number of conjugacy classes of parabolic elements). Hence there exists a positive number C such that for every element in this finite number of conjugacy classes, either $\lambda \mapsto \text{tr}^2 g_{\lambda}$ is constant, or there is a parameter $\lambda_1 \in \Lambda''$ such that $|\text{tr}^2 g_{\lambda_1} - t| \geq C$. On the other hand, for the elements $g \in G$ satisfying $|\lambda_{\max}(g_{\lambda_0})| > |t| + 4$, we have that $|\text{tr}^2 g_{\lambda_0} - t| > 1$. From this discussion and (24), we obtain the desired bound on the L^1 norm by passing to logarithms and applying Lemma 4.10 with $x_0 = \lambda_0$, $\Omega = \Lambda$, $\Omega' = \Lambda'$ and $\Omega'' = \Lambda''$. \square

4.3. Collisions between fixed points. Here we examine the distribution of another natural codimension 1 phenomenon in the bifurcation locus. For a pair of elements (g, h) in G , consider the subvariety in Λ defined by

$$F(g, h) = \{\lambda, \text{Fix}(g_{\lambda}) \cap \text{Fix}(h_{\lambda}) \neq \emptyset\}.$$

As before, if $F(g, g') = \Lambda$ we declare that $[F(g, g')] = 0$.

The associated equidistribution statement is the following.

Theorem 4.11. *Let (G, μ, ρ) be an admissible family of representations satisfying the exponential moment condition (8).*

Then for $(\mu^{\mathbb{N}} \otimes \mu^{\mathbb{N}})$ -a.e. $(\mathbf{g}, \mathbf{h}) \in (G^{\mathbb{N}})^2$, we have

$$\frac{1}{4n} [F(l_n(\mathbf{g}), l_n(\mathbf{h}))] \xrightarrow{n \rightarrow \infty} T_{\text{bif}}.$$

If furthermore one of the conditions i., ii. of Theorem 4.8 holds, then the convergence takes place in $L^1(\mu^{\mathbb{N}} \otimes \mu^{\mathbb{N}})$.

It is certainly possible to give estimates for the speed of convergence in the spirit of Theorem 4.12, but we omit this. Also, we may choose \mathbf{h} to be generic with respect to some other measure μ' on G , in which case, $\frac{1}{2n} [F(l_n(\mathbf{g}), l_n(\mathbf{h}))] \xrightarrow{n \rightarrow \infty} T_{\text{bif}} + T'_{\text{bif}}$, where T'_{bif} is the bifurcation current associated to (G, μ', ρ)

Proof. Recall from Lemma 2.3 that $F(g, h) = \{\lambda, \operatorname{tr}[g_\lambda, h_\lambda] = 2\}$. Notice that this allows us to properly define the multiplicity of $F(g, h)$. Passing to potentials, what we need to prove is that for $(\mu^\mathbb{N} \otimes \mu^\mathbb{N})$ -a.e. (\mathbf{g}, \mathbf{h}) ,

$$\frac{1}{4n} \log |\operatorname{tr}[l_n(\mathbf{g}), l_n(\mathbf{h})] - 2| \xrightarrow{n \rightarrow \infty} \chi(\lambda) \text{ in } L^1_{\text{loc}}.$$

Again for this we use Theorem 4.1, for $F(\gamma_1, \gamma_2) = \log |\operatorname{tr}[\gamma_1, \gamma_2] - 2|$. The plurisubharmonicity and *i.* are obvious, while *ii.* follows from Corollary A.5.

The proof of Theorem 4.8 shows that under one of the additional assumption *i.* or *ii.* of that theorem, for every $\Lambda' \Subset \Lambda$, $\mathbf{M}_{\Lambda'}([F(g, h)]) \leq C(\operatorname{length}(g) + \operatorname{length}(h))$. Therefore the second assertion of Theorem 4.11 follows from Proposition 4.6. \square

4.4. Speed of convergence. We now prove Theorem C.

Theorem 4.12. *Let (G, μ, ρ) be an admissible family of representations satisfying the exponential moment condition (8), and fix $t \in \mathbb{C}$.*

Suppose in addition that one of the following conditions holds:

- i. either Λ is an algebraic family of representations, defined over $\overline{\mathbb{Q}}$;*
- ii. or there is at least one geometrically finite representation in Λ .*

Then there exists a constant C such that for every test form ϕ

$$(25) \quad \left\langle \frac{1}{2n} \int [Z(g, t)] d\mu^n(g) - T_{\text{bif}}, \phi \right\rangle \leq C \frac{\log n}{n} \|\phi\|_{C^2}.$$

A few words about the proof: the machinery of Theorem 4.1, based on a compactness argument, does not allow for such an estimate, so the idea is to reprove Theorem 4.8 from scratch by using the quantitative results of Appendix A. The necessity to integrate with respect to \mathbf{g} is due to the fact that the estimate on $\delta(\rho(l_n(\mathbf{g})))$ given in Theorem A.1 is too sensitive to bifurcations to be made uniform in λ . As already said, a basic source of difficulty is that in general one cannot expect that for a given λ , $\frac{1}{2n} \log |\operatorname{tr}^2(\rho_\lambda(l_n(\mathbf{g}))) - t|$ converges to $\chi(\lambda)$ in $L^1(\mu^\mathbb{N})$. To control the size of the set of “exceptional” parameters where this convergence does not hold, we use volume estimates for sublevel sets of psh functions. As usual the notation C stands for a “constant” which may change from line to line, but does not depend on n .

Proof. Fix a finite set $\{g_1, \dots, g_k\}$ of generators and let $B_G(\text{id}, R) \subset G$ be the set of elements of length at most R .

Assume first for simplicity that μ has finite support. To prove the desired estimate we work with potentials, so as before let $u(\lambda, g) = \log |\operatorname{tr}^2(g_\lambda) - t|$. Let $\mathcal{P}'_t = \bigcup_{\{s, |s-t| \leq 1\}} \mathcal{P}_s$ be the set of g such that $\operatorname{tr}^2(g_\lambda)$ is a constant close to t . By the Large Deviations Theorem for the traces (Corollary A.2) $\mu^n(\mathcal{P}'_t)$ decreases to zero exponentially fast. We define $u_n(\lambda)$ by the formula

$$u_n(\lambda) = \frac{1}{2n} \int_{G \setminus \mathcal{P}'_t} u(\lambda, g) d\mu^n(g).$$

This is a psh potential of $\frac{1}{2n} \int [Z(g, t)] d\mu^n(g)$. Now to prove (25) it is enough to show that if $\Lambda' \subset \Lambda$ is a relatively compact open subset,

$$(26) \quad \|u_n - \chi\|_{L^1(\Lambda')} = O\left(\frac{\log n}{n}\right).$$

Assume that the condition *ii.* of Theorem 4.12 holds. We know from Lemma 4.9 that $\|u(\cdot, g)\|_{L^1(\Lambda')} \leq C \text{length}(g)$. Using standard estimates for the volume of sublevel sets of psh functions, we can control the volume of the set of representations possessing an element with trace too close to t .

Lemma 4.13. *Assume that the condition *ii.* of Theorem 4.12 is satisfied. Fix a relatively compact open subset $\Lambda' \subset \Lambda$, and a positive constant A . Then if $B > 0$ is large enough the volume of the open set*

$$V_n = \left\{ \lambda \in \Lambda' \text{ s.t. there exists } g \notin \mathcal{P}_t \text{ of length } \leq An \text{ with } |\text{tr}^2(g_\lambda) - t| < e^{-Bn^2} \right\}$$

is exponentially small in n .

Proof. It is no loss of generality to assume that $\Lambda' \Subset \Omega \Subset \Lambda$, where Ω is biholomorphic to (and viewed as) the ball $B(0, 1)$ in $\mathbb{C}^{\dim(\Lambda)}$. Fix $g \in G$ and let

$$\tilde{u}(\lambda, g) = \frac{1}{\text{length}(g)} u(\lambda, g) = \frac{1}{\text{length}(g)} \log |\text{tr}^2 g_\lambda - t|.$$

By Lemma 4.9, the family of psh functions $\tilde{u}(\cdot, g)$ is relatively compact in $L^1(\Lambda')$, so by Lemma 4.10, there exist constants M and A independent of g such that for every g , $\sup_{B(0,1)} \tilde{u}(\cdot, g) \geq M$ and there exists a point $x_0 \in B(0, 1/2)$ such that $\tilde{u}(x_0, g) \geq -A$.

By [H1, Theorem 4.4.5] (combined with Lemma 4.10) there exist constants c_1 and c_2 such that $\int \exp(-c_1 \tilde{u}(\cdot, g)) \leq c_2$ uniformly in g . Thus by the Markov inequality, there exists a constant a such that for every $s > 0$,

$$\text{vol}(\{\lambda \in \Lambda', \tilde{u}(\lambda, g) < -s\}) \leq C e^{-as}.$$

In this equation, we put $s = \frac{Bn^2}{\text{length}(g)}$, and infer that there exists a constant C such that if $\text{length}(g) \leq An$,

$$\text{vol}\left(\left\{\lambda \in \Lambda', |\text{tr}^2(g_\lambda) - t| < e^{-Bn^2}\right\}\right) \leq C \exp\left(-\frac{aBn}{A}\right).$$

To finish the proof, it is enough to sum this estimate over all words of length $\leq An$. Since the number of such elements is at most exponential in n (bounded by $Ce^{C(A)n}$), we conclude that if B is large enough (i.e. $B > AC(A)/a$), $\text{vol}(V_n)$ is exponentially small. \square

Remark 4.14. A similar estimate was proven in $\text{SU}(2)$ by Kaloshin and Rodnianski in [KR], by a different method. It appears that the use of pluripotential theoretic tools leads to a short proof of their result (see Remark 4.18 below).

Let us now assume that *i.* holds. To estimate the L^1 norm of $u(\cdot, g)$, we use the results of Appendix B.

Lemma 4.15. *Under the assumption *i.* of Theorem 4.12, for every relatively compact open subset $\Lambda' \subset \Lambda$ there exists a constant $\delta > 0$ such that for every $g \in G$, if $\lambda \mapsto \text{tr}^2(g_\lambda)$ is not constant, then*

$$\sup_{\lambda \in \Lambda'} |\text{tr}^2(g_\lambda) - t| \geq \delta^{\text{length}(g) \log(\text{length}(g))}.$$

Proof. Recall that Λ can be viewed as an open subset of an affine subvariety in \mathbb{C}^{9k} (k is the number of generators). If $g \in G$ is any element, then as before $\text{tr}^2(g)$ is the restriction to Λ of a polynomial in the $a_{i,j}$ (the matrix coefficients corresponding to the generator g_i) of

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degree $O(\text{length}(g))$, with integer coefficients. Furthermore, there exists a constant D such that each of these coefficients is bounded by $D^{\text{length}(g)}$. The estimate that we seek is now a direct consequence of Corollary B.2. \square

We have a version of Lemma 4.13 in this context.

Lemma 4.16. *Assume that condition i. of Theorem 4.12 holds. Fix a relatively compact open subset $\Lambda' \subset \Lambda$, and a positive constant A . Then if $B > 0$ is large enough the volume of the open set*

$$V_n = \left\{ \lambda \in \Lambda' \text{ s.t. there exists } g \notin \mathcal{P}'_t \text{ of length } \leq An \text{ with } |\text{tr}^2(g_\lambda) - t| < e^{-Bn^2 \log n} \right\}$$

is exponentially small in n .

Proof. It follows from the previous lemma and (24) that there exist a constant C such that for every $g \in G$,

$$(27) \quad \|u(\cdot, g)\|_{L^1(\Lambda')} \leq C \text{length}(g) \log(\text{length}(g)),$$

so we simply put $\tilde{u}(\lambda, g) = \frac{1}{\text{length}(g) \log(\text{length}(g))} u(\lambda, g)$ and argue as in Lemma 4.13. \square

We now resume the proof of the theorem. The less favorable situation is when *i.* holds, so let us put ourselves in this case. Fix a constant A in Lemma 4.16 such that $\text{Supp}(\mu^n) \subset B_G(\text{id}, An)$, and a corresponding constant B . Then, with notation as in the lemma, V_n has exponentially small volume. To prove that $\|u_n - \chi\|_{L^1(\Lambda')} = O\left(\frac{\log n}{n}\right)$, we will proceed in two steps: first show that this estimate holds pointwise outside V_n , by using Theorem A.1, and then use general facts on psh functions to get a global L^1 bound.

So first fix a parameter $\lambda \notin V_n$. As in Theorem A.1, let $\delta(g_\lambda)$ be the distance between the fixed points of g_λ . Let $\varepsilon_n = n^{-\alpha}$, where $\alpha > 0$ is a constant to be fixed later, and let

$$E_n^1 = \{g \in G, \delta(\rho_\lambda(g)) \geq \varepsilon_n\}.$$

By Theorem A.1, $\mu^n(E_n^1) \geq 1 - C\varepsilon_n^K$ for some K .

Let now $\chi_n(\lambda) = \frac{1}{n} \int_G \log \|g_\lambda\| d\mu^n(g)$. We claim that $\|\chi_n - \chi\|_{L^\infty} = O\left(\frac{1}{n}\right)$, in which case it will be enough to prove that $|u_n(\lambda) - \chi_n(\lambda)| = O\left(\frac{\log n}{n}\right)$. Indeed we know from Proposition 3.8 that if $z_0 \in \mathbb{P}^1$ is fixed, then $\|\chi_{n,z_0} - \chi\|_{L^\infty} = O\left(\frac{1}{n}\right)$, where $\chi_{n,z_0}(\lambda) = \frac{1}{n} \int \log \|g_\lambda(z_0)\| d\mu^n(g) = \frac{1}{n} \int \log \frac{\|g_\lambda(Z_0)\|}{\|Z_0\|} d\mu^n(g)$. Moreover it is clear from the proof of that lemma that the $O(\cdot)$ is uniform with respect to z_0 . Thus to get our claim it is enough to integrate with respect to z_0 and apply Lemma 2.2.

Let E_n^2 be the set of those $g \in E_n^1$ such that moreover $g \notin \mathcal{P}'_t$ and $\left|\frac{1}{n} \log \|g_\lambda\| - \chi(\lambda)\right| \leq \chi(\lambda)/2$. Then $\mu^n(E_n^1 \setminus E_n^2)$ is exponentially small and accordingly, $\mu^n(E_n^2) \geq 1 - C\varepsilon_n^K$.

We split $u_n - \chi_n$ as

$$\begin{aligned} u_n(\lambda) - \chi_n(\lambda) &= \frac{1}{n} \int_{E_n^2} \left(\frac{1}{2} \log |\text{tr}^2(g_\lambda) - t| - \log \|g_\lambda\| \right) d\mu^n(g) + \\ &\quad + \frac{1}{n} \int_{(E_n^2)^c \setminus \mathcal{P}_t} \frac{1}{2} \log |\text{tr}^2(g_\lambda) - t| d\mu^n(g) - \frac{1}{n} \int_{(E_n^2)^c} \log \|g_\lambda\| d\mu^n(g) \end{aligned}$$

To estimate the first integral, we use Lemma 2.1. Indeed, since $\delta(g_\lambda) \geq \varepsilon_n = n^{-\alpha}$ we see that if $g \in E_n^2$, $\text{tr}^2(g_\lambda)$ is of order of magnitude $e^{n\chi(\lambda)}$ so $\frac{1}{n} \log |\text{tr}^2(g_\lambda) - t| =$

$\frac{1}{n} \log |\operatorname{tr}^2(g_\lambda) - 4| + o\left(\frac{1}{n}\right)$, and $\frac{1}{2} \log |\operatorname{tr}^2(g_\lambda) - t| - \log \|g_\lambda\| \sim (\log \delta(g_\lambda)) = O(\log n)$. We deduce that this first integral is $O\left(\frac{\log n}{n}\right)$.

The third integral is bounded by $C\mu^n((E_n^2)^c) = C\varepsilon_n^K = Cn^{-\alpha K}$ which is $O(n^{-2})$ if α is large enough (recall that K does not depend on α).

Finally, to estimate the second integral, we use the fact that $\lambda \notin V_n$. From this we infer that $|\log |\operatorname{tr}^2(g_\lambda) - t|| \leq Bn^2 \log n$, thus the integral is bounded by $\frac{1}{n} \mu^n((E_n^2)^c) Bn^2 \log n = O(n^{1-\alpha K} \log n)$, which again is $O(n^{-2})$ for large α .

It is clear that all the $O(\cdot)$ appearing in the above reasoning are uniform for $\lambda \in \Lambda' \setminus V_n$. Thus at this point we know that u_n is a psh function, bounded from above (by (24)), with $\|u_n\|_{L^1(\Lambda')} \leq C \log n$ (by (27); notice that the $\log n$ is superfluous under *ii.*) and $\|u_n - \chi\|_{L^\infty(\Lambda' \setminus V_n)} = O\left(\frac{\log n}{n}\right)$.

To complete the proof of the desired estimate (26) in case μ has finite support, it remains to show that $\|u_n - \chi\|_{L^1(V_n)} = O\left(\frac{\log n}{n}\right)$, which is done in the following easy lemma.

Lemma 4.17. *Under the above assumptions and notation, $\|u_n - \chi\|_{L^1(V_n)}$ is exponentially small in n .*

Proof. Recall from Lemmas 4.13 and 4.16 that $\operatorname{vol}(V_n)$ is exponentially small. By boundedness of χ it follows that $\|\chi\|_{L^1(V_n)}$ is exponentially small.

To control $\|u_n\|_{L^1(V_n)}$, put $\tilde{u}_n = \frac{1}{\log n} u_n$. Since (\tilde{u}_n) is bounded in L^1 , it is no loss of generality to assume that $\tilde{u}_n \leq 0$, then as before there exists a constant a such that $\operatorname{vol}(\{\tilde{u}_n < -M\}) \leq e^{-aM}$. Thus we simply write

$$\int_{V_n} |\tilde{u}_n| = \int_{V_n \cap \{\tilde{u}_n < -M\}} |\tilde{u}_n| + \int_{V_n \cap \{\tilde{u}_n \geq -M\}} |\tilde{u}_n| \leq \int_{\{\tilde{u}_n < -M\}} |\tilde{u}_n| + M \operatorname{vol}(V_n),$$

and use the coarea formula

$$\int_{\{\tilde{u}_n < -M\}} |\tilde{u}_n| = \int_0^M \operatorname{vol}(\tilde{u}_n < -t) dt + \int_M^{+\infty} \operatorname{vol}(\tilde{u}_n < -t) dt = O(Me^{-aM})$$

to deduce that $\int_{V_n} |\tilde{u}_n| = O(Me^{-aM} + M \operatorname{vol}(V_n))$. To conclude that $\int_{V_n} |\tilde{u}_n|$, whence $\int_{V_n} |u_n|$, is exponentially small, it suffices to pick $M = n$. \square

It remains to treat the case where the support of μ is infinite. Recall that we assume that μ satisfies (8). We adapt the proof by using the exponential moment condition to show that μ almost behaves like a measure with finite support, and obtain exponentially decaying estimates for the resulting errors. Again we work under the less favorable assumption *i.*

It is an easy consequence of the moment condition that for a sufficiently large constant A , $\mu^n(B_G(\operatorname{id}, An)^c)$ and more generally $\int_{B_G(\operatorname{id}, An)^c} \operatorname{length}(g) d\mu^n(g)$ tend to zero exponentially fast. Indeed, let $I = \int_G \exp(\tau \operatorname{length}(g)) d\mu(g)$ which is finite by assumption. Subadditivity of the length implies that $\int_G \exp(\tau \operatorname{length}(g)) d\mu^n(g) \leq I^n$, therefore, by the Markov inequality, for every $s > 0$, $\mu^n(\{g, \operatorname{length}(g) \geq s\}) \leq \exp(-\tau s) I^n$.

We then infer that

$$\begin{aligned} \int_{B_G(\text{id}, An)^c} \text{length}(g) \log(\text{length}(g)) d\mu^n(g) &= \sum_{k=An+1}^{\infty} k(\log k) \mu^n(\{g, \text{length}(g) = k\}) \\ &\leq I^n \sum_{k=An+1}^{\infty} k(\log k) e^{-\tau k} \end{aligned}$$

which decreases exponentially if A is sufficiently large.

Now, recall from Lemma 4.9 that $\|u(\cdot, g)\|_{L^1(\Lambda')} = O(\text{length}(g))$. A first consequence is that the sequence (u_n) is bounded in $L^1(\Lambda')$. With notation as before, it is enough to show that $\|u_n - \chi_n\|_{L^1(\Lambda')} = O\left(\frac{\log n}{n}\right)$. For a constant A as just above, decompose u_n as

$$u_n = \frac{1}{2n} \int_{(G \setminus \mathcal{P}'_t) \cap B(\text{id}, An)} u(\cdot, g) d\mu^n(g) + \frac{1}{2n} \int_{(G \setminus \mathcal{P}'_t) \cap B(\text{id}, An)^c} u(\cdot, g) d\mu^n(g) =: u_n^1 + u_n^2,$$

and similarly for χ_n , and write $u_n - \chi_n = (u_n^1 - \chi_n^1) + (u_n^2 - \chi_n^2)$. The first part of the proof shows that $\|u_n^1 - \chi_n^1\|_{L^1(\Lambda')} = O\left(\frac{\log n}{n}\right)$ while the above considerations imply that $\|u_n^2 - \chi_n^2\|_{L^1(\Lambda')}$ decreases to zero exponentially fast. The proof is complete. \square

Remark 4.18. The proof actually says more. A measure m on Λ is said to be *moderate* [DS2] if for any $\Lambda'' \Subset \Lambda' \Subset \Lambda$, there exist constants $C, \alpha > 0$ such that if u is a psh function with $\|u\|_{L^1(\Lambda')} \leq 1$, then for every $s > 0$,

$$m(\{\lambda \in \Lambda'', u(\lambda) < -s\}) \leq C e^{-\alpha s}.$$

An obvious adaptation of the proof shows that $\|u_n - \chi\|_{L^1_{\text{loc}}(m)} = O\left(\frac{\log n}{n}\right)$ for any moderate measure m .

This observation has several interesting consequences. Dinh and Sibony [DS2] showed that if T is a (1,1) current with Hölder continuous potentials, then its trace measure σ_T , and more generally that of its successive exterior powers $T \wedge \cdots \wedge T$ are moderate. As a consequence, if we let $Z_n = \frac{1}{2n} \int [Z(g, t)] d\mu^n(g)$, then for every $q \leq \dim(\Lambda) - 1$, if ϕ is a test form of the right dimension,

$$\left\langle Z_n \wedge T_{\text{bif}}^q - T_{\text{bif}}^{q+1}, \phi \right\rangle \leq C \frac{\log n}{n} \|\phi\|_{C^2}.$$

Such an estimate might prove useful (as Proposition 3.8 was for $q = 1$) when trying to characterize $\text{Supp}(T_{\text{bif}}^q)$ (see §5.2.2).

It is a classical fact that the area measure on a totally real submanifold of maximal dimension is moderate –this may also easily be deduced from [DS2]. This applies in particular to the area measure on $\text{SU}(2) \subset \text{SL}(2, \mathbb{C})$. Thus, arguing exactly as in Lemma 4.13, we can recover the original Kaloshin-Rodnianski estimate [KR]: there exists a constant $B > 0$ such that the volume of the set of $(u, v) \in \text{SU}(2)$ with the property that there exists a word w of length n in the free group \mathbb{F}_2 such that $\text{dist}(w(u, v), \text{id}) < \exp(-Bn^2)$, is exponentially small (recall that if $w \in \text{SU}(2)$, $\|w - \text{id}\|^2 \asymp |\text{tr}(w) - 2|$). For this, we use the fact that *ii.* holds in this context, that is, the free group \mathbb{F}_2 admits geometrically finite representations into $\text{SL}(2, \mathbb{C})$, e.g. Schottky subgroups. Notice that the same applies to $\text{U}(n) \subset \text{GL}(n, \mathbb{C})$ (resp. $\text{SU}(n) \subset \text{SL}(n, \mathbb{C})$), and to free groups with arbitrary many generators.

4.5. Motion of fixed points in $\Lambda \times \mathbb{P}^1$. Keeping notation as in §3.2, for g in $G \setminus \text{id}$, we let $\text{Fix}(\widehat{g})$ be the hypersurface in $\Lambda \times \mathbb{P}^1$ defined by the equation $\{(\lambda, z), g_\lambda(z) = z\}$ (counted with its multiplicity). Notice that if λ_0 is such that $g_{\lambda_0} = \text{id}$, then $\{\lambda_0\} \times \mathbb{P}^1 \subset \text{Fix}(\widehat{g})$.

Theorem 4.19. *Let (G, μ, ρ) be an admissible family of representations of G into $\text{PSL}(2, \mathbb{C})$ satisfying the exponential moment condition (8).*

Then for $\mu^\mathbb{N}$ -a.e. $\mathbf{g} \in G^\mathbb{N}$, the sequence of currents of bidegree $(1, 1)$ $\frac{1}{n} [\text{Fix}(\widehat{l_n(\mathbf{g})})]$ on $\Lambda \times \mathbb{P}^1$ converges to $\pi_1^(T_{\text{bif}})$.*

If furthermore, one of the additional assumptions i. and ii. of Theorem 4.8 is satisfied, then the convergence takes place in $L^1(\mu^\mathbb{N})$.

Again we may interpret this by saying that $\lambda_0 \in \text{Supp}(T_{\text{bif}})$ iff for every neighborhood U of λ_0 the average volume of $\text{Fix}(\widehat{l_n(\mathbf{g})}) \cap \pi_1^{-1}(U)$ grows linearly with n .

Proof. The proof is similar to that of Theorem 3.5 so we shall be brief. Again, the result is local on Λ so we may assume it is a ball, and we decompose the Kähler form in $\Lambda \times \mathbb{P}^1$ as $\widehat{\omega} = \pi_1^*\omega + \pi_2^*\omega_{\mathbb{P}^1}$.

For every $g \in G$, $\pi_1 : \text{Fix}(\widehat{g}) \rightarrow \Lambda$ is a dominant mapping of degree at most 2, with possibly some exceptional fibers corresponding to parameters where $\lambda_0 = \text{id}$. In any case we infer that $\langle \widehat{T}_n, \pi_1^*\omega^k \rangle \rightarrow 0$. Thus, again, what we need to analyze is pairings of the form $\langle \widehat{T}_n, \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle = \langle (\pi_1)_*(\widehat{T}_n \wedge \pi_2^*\omega_{\mathbb{P}^1}), \phi \rangle$, where ϕ is a $(k-1, k-1)$ test form on Λ .

Let $I(g)$ (resp. $P(g)$) be the subvariety of Λ defined by $I(g) = \{\lambda, g_\lambda = \text{id}\}$ (resp. $P(g) = \{\lambda, g_\lambda \text{ is parabolic}\}$). Consider an open subset $\Omega \subset \Lambda$ disjoint from $I(g)$, so that $\pi_1|_{\text{Fix}(\widehat{g}) \cap \pi_1^{-1}(\Omega)}$ is a branched cover of degree 1 (in the case of a persistently parabolic element) or 2 (in the other cases).

Suppose first that g is not persistently parabolic, and pick a ball U where g_λ is never parabolic. In this case, $\text{Fix}(\widehat{g}) \cap \pi_1^{-1}(U)$ consists of two graphs $\text{Fix}_1(\widehat{g})$ and $\text{Fix}_2(\widehat{g})$ over U corresponding to the two fixed points of g_λ . Let us denote these by $f_i(\lambda)$, $i = 1, 2$. As in (15) we obtain that

$$\langle [\text{Fix}(\widehat{g})], \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle = \langle [\text{Fix}_1(\widehat{g})] + [\text{Fix}_2(\widehat{g})], \pi_2^*\omega_{\mathbb{P}^1} \wedge \pi_1^*\phi \rangle = \int_\Lambda ((f_1)^*\omega_{\mathbb{P}^1} + (f_2)^*\omega_{\mathbb{P}^1}) \wedge \phi$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; the coefficients are defined only up to sign, but this does not affect the foregoing formulas. For the matter of computation we may assume that c_λ never vanishes in U , so that f_1 and f_2 take their values in a fixed affine chart $\mathbb{C} \subset \mathbb{P}^1$. We then obtain that

$$\begin{aligned} (f_1)^*\omega_{\mathbb{P}^1} + (f_2)^*\omega_{\mathbb{P}^1} &= dd^c \log \left(1 + |f_1|^2 \right)^{\frac{1}{2}} + dd^c \log \left(1 + |f_2|^2 \right)^{\frac{1}{2}} \\ &= dd^c \log \left(|b_\lambda|^2 + |c_\lambda|^2 + \frac{|d_\lambda - a_\lambda|^2 + |\text{tr}^2(g_\lambda) - 4|}{2} \right)^{\frac{1}{2}}. \end{aligned}$$

The last equality is in turn also valid when c_λ vanishes. Let $v(\lambda, g)$ be the argument of the dd^c in the last line.

Assume that $P(g) \cap \Omega$ is not empty (recall that by assumption $I(g) \cap \Omega = \emptyset$). The function $v(\lambda, g)$ is locally bounded near $P(g)$ so $dd^c v(\cdot, g)$ gives no mass to $P(g)$. Likewise, $[\text{Fix}(\widehat{g})]$ gives no mass to $\pi^{-1}(P(g) \cap \Omega)$. Therefore we conclude that $v(\cdot, g)$ is a potential of $(\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^*\omega_{\mathbb{P}^1})$ throughout Ω .

It is straightforward to check that the same holds when g_λ is persistently parabolic.

At this point we know that $v(\cdot, g)$ is a potential of $(\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^* \omega_{\mathbb{P}^1})$ outside $I(g)$. We claim that this is actually true everywhere on Λ . Notice first that $v(\cdot, g)$ is a well-defined psh function, with poles on $I(g)$. Let Σ be an irreducible component of $I(g)$. The extension of $v(\cdot, g)$ as a potential of $(\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^* \omega_{\mathbb{P}^1})$ over Σ is immediate when $\text{codim}(\Sigma) > 1$, since since neither $dd^c v(\cdot, g)$ nor $(\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^* \omega_{\mathbb{P}^1})$ would carry any mass on Σ in this case. So we can suppose that $\text{codim}(\Sigma) = 1$, and, slicing by 1-dimensional submanifolds, we may further assume that $\dim(\Lambda) = 1$ and $\Sigma = \{\lambda_0\}$. In this case, the measure $(\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^* \omega_{\mathbb{P}^1})$ has an atom of multiplicity m at λ_0 , where m is the generic multiplicity of $\text{Fix}(\widehat{g})$ along $\pi_1^{-1}(\Sigma)$. Let us compute m : we are looking at the multiplicity of the root λ_0 of the equation $c_\lambda z^2 + (d_\lambda - a_\lambda)z + b_\lambda = 0$ for generic z , at a parameter λ_0 where $b_{\lambda_0} = c_{\lambda_0} = d_{\lambda_0} - a_{\lambda_0} = 0$. We infer that

$$m = \min(\text{mult}_{\lambda_0}(b_\lambda), \text{mult}_{\lambda_0}(c_\lambda), \text{mult}_{\lambda_0}(d_\lambda - a_\lambda)).$$

To prove that $dd^c v(\cdot, g) = (\pi_1)_*([\text{Fix}(\widehat{g})] \wedge \pi_2^* \omega_{\mathbb{P}^1})$, it is enough to show that $dd^c v(\cdot, g)$ also admits an atom of multiplicity m at λ_0 . Equivalently, we need to show that $v(\cdot, g)$ has a logarithmic pole of order m at λ_0 , which is clear from the formula defining v and the observation that $\text{mult}_{\lambda_0}(\text{tr}^2(g_\lambda) - 4) \geq 2m$.

By definition of $v(\cdot, g)$, it is clear that there exists a constant C independent of g such that

$$(28) \quad \min\left(\frac{1}{2} \log |\text{tr}^2(g_\lambda) - 4|, \log \|g_\lambda\|\right) - C \leq v(\lambda, g) \leq \log \|g_\lambda\| + C.$$

As usual, we conclude from Theorem 4.1 that $\frac{1}{n}[\text{Fix}(\widehat{l_n(\mathbf{g})})]$ converges to $\pi_1^*(T_{\text{bif}})$.

Finally, we leave the reader check that under each of the assumptions *i.* or *ii.* of Theorem 4.8, (28) shows that the mass of $\frac{1}{n}[\text{Fix}(\widehat{l_n(\mathbf{g})})]$ is locally controlled by $\text{length}(g)$. So the second assertion of the theorem follows from Proposition 4.6. \square

5. FURTHER COMMENTS

5.1. Canonical bifurcation currents. Let S be a Riemann surface of genus $g \geq 2$ and $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ the set of representations of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$. Take a holomorphic family $\Lambda \subset \text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{C}))$ made of non-elementary representations. Our purpose in this paragraph is to outline the construction of a canonical bifurcation current on Λ , depending only on the Riemann surface structure. The details will appear in a subsequent paper [DD].

The idea consists in replacing the *discrete* random walk on $\pi_1(S)$ by a *continuous* Markov process on S : the Brownian motion with respect to a conformal metric. This defines a Lyapunov exponent, very much in the spirit of [DK]. The induced function on Λ is well-defined up to a multiplicative constant, because of the conformal invariance of the Brownian motion.

To be more precise, denote by \widetilde{S} the universal cover of S . Given a representation $\rho \in \Lambda$, consider the flat \mathbb{P}^1 -bundle over S with monodromy ρ , that we denote by X . Recall that it is obtained by taking the quotient of $\widetilde{S} \times \mathbb{P}^1$ by the diagonal action of $\pi_1(S)$ (defined by $\gamma(x, z) = (\gamma x, \rho(\gamma)z)$). If $x \in S$, we denote the fiber of the bundle over x by X_x . Observe that to any oriented continuous path γ with endpoints x and y corresponds a holonomy map h_γ from X_x to X_y , obtained by lifting γ as a family of continuous paths in the flat sections.

A spherical metric $\|\cdot\|$ being given on the \mathbb{P}^1 -fibers, for every path $\gamma : [0, \infty) \rightarrow S$ we may consider the limit

$$(29) \quad \chi(\gamma) = \lim_{t \rightarrow \infty} \frac{\log \|h_{\gamma|_{[0,t]}}\|}{t}.$$

If γ is a generic Brownian path, the limit in (29) indeed exists and only depends on the conformal metric and the representation (but not on γ). As already said, two different conformal metrics give rise to Lyapunov exponent functions on Λ that differ only by a multiplicative constant. To specify this constant it is enough to fix the metric as being the Poincaré metric of constant curvature -1 .

It is not difficult to convince oneself that the function χ on Λ is psh. We can thus define a bifurcation current on Λ , depending only on the complex structure on S , by the formula $T_{\text{bif}} = dd^c \chi$. What is less obvious is that there actually exists a measure on $\pi_1(S)$, possessing exponential moments, and such that T_{bif} is the associated bifurcation current on Λ . In particular we have that:

Theorem 5.1. *The support of T_{bif} is the bifurcation locus.*

It is also possible to state equidistribution theorems involving summations over the set of closed geodesics on S .

Here is a situation where these ideas naturally apply: consider the set of complex projective structures over a Riemann surface S , compatible with its complex structure. This is an affine space of dimension $3g - 3$, admitting a distinguished point, namely the projective structure obtained by viewing S as a quotient of the unit disk (see [Dum] for an introductory text on this). The so-called (and much studied) *Bers slice* of Teichmüller space is the connected component of this point in the stability locus. A projective structure induces a *monodromy representation* (which is always non-elementary and defined only up to conjugacy) so the above discussion applies and we conclude that *the space of projective structures on S admits a canonical bifurcation current*. We also show in [DD] that the Lyapunov exponent function is *constant* on the Bers slice. Through the Sullivan dictionary (as extended in [Mc]), this corresponds to the theorem that the Lyapunov exponent of a monic polynomial of degree d with connected Julia set is equal to $\log d$.

5.2. Open questions.

5.2.1. Arguably the most important question left open in the paper is: how does T_{bif} depend on μ ? For instance, are the bifurcation currents mutually singular/absolutely continuous when μ varies?

In this context it may be interesting to note that if $\text{Supp}(\mu)$ is finite, then $\chi(\mu)$ is a real analytic function of the transition probabilities for a fixed representation [Pe].

Here is a related question: assume that Λ is the character variety of representations of G into $\text{PSL}(2, \mathbb{C})$. How does the outer automorphism group $\text{Out}(G)$ act on the bifurcation currents? Is it possible to find a measure μ so that $\text{Out}(G)$ preserves the measure class induced by T_{bif} ?

5.2.2. For spaces of rational maps, the description of the exterior powers of T_{bif} is an important theme, with again some emphasis on the characterization of their supports and equidistribution theorems [BB1, BB2, BB3, DF, Du, BE, Ga]. The underlying ideology is that

$\text{Supp}(T_{\text{bif}}^k)$, for $1 \leq k \leq \dim(\Lambda)$ should define a dynamically meaningful filtration of the bifurcation locus.

It is also natural to investigate this question in our context. However, it seems that the supports of T_{bif}^k do not give any new information here. To be precise, assume that $\dim(\Lambda) \geq 2$ and that different representations in Λ are never conjugate (i.e. Λ is a subset of the character variety). Then *we conjecture that for every $k \leq \dim(\Lambda)$, $\text{Supp}(T_{\text{bif}}^k) = \text{Bif}$.*

Here is some evidence for this: let $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$, $t = 4\cos^2(\theta)$ and look at the varieties $Z(g, t)$. Since for $\lambda \in Z(g, t)$, ρ_λ is not discrete, the bifurcation locus of $\{\rho_\lambda, \lambda \in Z(g, t)\}$ is equal to $Z(g, t)$. Hence $\text{Supp}(T_{\text{bif}} \wedge [Z(g, t)]) = Z(g, t)$, which by Theorem 4.7 makes the equality $\text{Supp}(T_{\text{bif}}^2) = \text{Supp}(T_{\text{bif}})$ plausible (see also Remark 4.18).

Notice that the currents constructed by Cantat in [Ca] as natural invariant currents under holomorphic automorphisms of the character variety, have zero self-intersection.

5.2.3. We know that the normalized currents of integration over $Z(l_n(\mathbf{g}), 4)$ are equidistributed towards T_{bif} . Now, the parameters in $Z(l_n(\mathbf{g}), 4)$ can be of two types: parabolic or identity. Is there a dominant one? One might guess that parabolic parameters prevail.

5.2.4. Do our results lead to efficient algorithms for producing computer pictures of stability/bifurcation loci? It is often a delicate issue in this type of problems to find numerical criteria deciding whether a representation is discrete. Here, given a group endowed with a measure and a one-dimensional family of representations, one may simply try to plot numerically the Lyapunov exponent function and look for regions where it is harmonic.

Another approach would be to use Theorem 4.7 to obtain an approximation of the bifurcation locus by plotting the solutions to $\text{tr}^2(\rho_\lambda(g)) = 4$ for (a small number of) large random elements $g = g_n \cdots g_1$.

5.2.5. Can T_{bif} be described more precisely in some particular families (say, one-dimensional, so that T_{bif} simply becomes a measure)? Are there measures μ for which T_{bif} is absolutely continuous with respect to Lebesgue measure? Are there measures for which T_{bif} gives some mass to representations with values in $\text{PSL}(2, \mathbb{R})$? Is the boundary of the bifurcation locus always of zero measure?

APPENDIX A. ESTIMATES ON THE DISTANCE BETWEEN FIXED POINTS AND APPLICATIONS

In this section we give some refinements of the classical results on random matrix products that we presented in §2.3. Similar results have been recently proven in a much more general setting by Aoun [Ao]. In our context the proofs simplify greatly, so we include them for convenience –besides, we need estimates slightly different from his.

If $\gamma \in \text{PSL}(2, \mathbb{C})$ we denote by $\delta(\gamma)$ the distance between its fixed points (on \mathbb{P}^1). Our first purpose is to show that on a set of large $\mu^\mathbb{N}$ measure, $\rho(l_n(\mathbf{g}))$ is a loxodromic element with well-separated fixed points (recall that for $\mathbf{g} \in G^\mathbb{N}$, $l_n(\mathbf{g}) = g_n \cdots g_1$).

Theorem A.1. *Let (G, μ, ρ) be a non-elementary representation satisfying the exponential moment condition (7).*

Then there exists a constant K such that if ε_n is one of the sequences $cn^{-\alpha}$ (for $c, \alpha > 0$), or $\exp(-\gamma n)$ (for $0 < \gamma < \gamma_0$ where γ_0 is an explicit constant), then n for large enough,

$$\mathbb{P}(\delta(\rho(l_n(\mathbf{g}))) < \varepsilon_n) \leq \varepsilon_n^K.$$

As a first consequence, we obtain the following large deviation estimate in Theorem 2.10

Corollary A.2. *Let (G, μ, ρ) be a non-elementary representation satisfying the exponential moment condition (7).*

Then for every positive real number ε , the probability

$$\mathbb{P} \left(\left| \frac{1}{n} \log |\operatorname{tr} \rho(l_n(\mathbf{g}))| - \chi \right| > \varepsilon \right)$$

decreases to zero exponentially fast when n tends to infinity.

Of course this implies Theorem 2.10, by applying the Borel-Cantelli lemma.

Proof of the corollary. Under these assumptions, the Large Deviation Theorem holds for the distribution of the values of $\frac{1}{n} \log \|l_n(\mathbf{g})\|$ [BL, §V.6]. Thus if ε' is small, the probability $\mathbb{P} \left(\left| \frac{1}{n} \log \|l_n(\mathbf{g})\| - \chi \right| > \varepsilon' \right)$ is exponentially small in n .

From Lemma 2.1 we know that when $\|\gamma\|$ is large enough,

$$\frac{1}{2} \log |\operatorname{tr}^2 \gamma - 4| = \log \|\gamma\| + \log \delta(\gamma) + O(1).$$

So the result follows by taking $\varepsilon_n = \exp(-\varepsilon'' n)$ in Theorem A.1 where $\varepsilon', \varepsilon'' > 0$ are such that $\varepsilon' + \varepsilon'' < \varepsilon$. \square

Proof of Theorem A.1. We start with a lemma.

Lemma A.3. *Let (ε_p) be as in the statement of Theorem A.1, then there exist constants $C, K > 0$ such that for any pair (x_0, y_0) of points of \mathbb{P}^1 , and any integer p*

$$\mathbb{P}(\rho(l_p(\mathbf{g}))(x_0) \in B(y_0, \varepsilon_p)) \leq C \varepsilon_p^K.$$

(here $B(y, r)$ is the ball of center y and radius r with respect to the spherical distance.)

Proof. Recall that there are constants $C, \alpha, \beta > 0$ such that for every integer p we have

$$(30) \quad \|P^p - N\|_\alpha \leq C e^{-\beta p}.$$

In addition, it is known that the stationary measure has Hölder regularity, in the sense that there are constants $C, \eta > 0$ such that $\nu(B(x, r)) \leq C r^\eta$ for every $x \in \mathbb{P}^1$ and every radius $r > 0$ (see [BL, p.161])

Introduce a function $f : \mathbb{P}^1 \rightarrow [0, 1]$ such that

- f is identically 1 on the ball $B(y_0, \varepsilon_p)$,
- f vanishes outside $B(y_0, 2\varepsilon_p)$,
- f is Lipschitz with Lipschitz constant bounded by $\frac{1}{\varepsilon_p}$.

The existence of such a function is straightforward.

Observe that f is α -Hölder with $\|f\|_{C^\alpha}$ being bounded above by the Lipschitz norm, namely $\|f\|_{C^\alpha} \leq \varepsilon_p$. Moreover, we have

$$\int f d\nu \leq \nu(B(y_0, 2\varepsilon_p)) \leq C \varepsilon_p^\eta.$$

Applying (30) to this function, we get that

$$\mathbb{P}(\rho(l_p(\mathbf{g}))(x_0) \in B(y_0, \varepsilon_p)) \leq P^p f(x_0) \leq \int f d\nu + C e^{-\beta p} \|f\|_{C^\alpha},$$

and we conclude that

$$\mathbb{P}(\rho(l_p(\mathbf{g}))(x_0) \in B(y_0, \varepsilon_p)) \leq C \left(\varepsilon_p^\eta + \frac{e^{-\beta p}}{\varepsilon_p} \right).$$

If ε_n decreases sub-exponentially fast, this is smaller than $C\varepsilon_p^K$ for $K = \min(\eta, 1)$. If $\varepsilon_n = \exp(-\gamma n)$ the same holds as soon as $\gamma < \beta$. \square

Let $\varepsilon > 0$ be a real number which is small with respect to χ ($\varepsilon = \frac{1}{100} \min(\chi, 1)$ should be enough) and introduce the integer $m = \lfloor (1 - \varepsilon)n \rfloor$. We divide the composition $l_n(\mathbf{g})$ into two parts of respective lengths m and $n - m$: $l_n(\mathbf{g}) = (g_n \dots g_{n-m+1})(g_m \dots g_1)$. The first part will have the effect of making $\rho(l_n(\mathbf{g}))$ of large norm (approximately $e^{(\chi + O(\varepsilon))n}$) with high probability, while the second one will be used to separate its fixed points by a distance of the order of magnitude of ε_n .

The quantity

$$\mathbb{P}\left(\left|\frac{1}{m} \log \|\rho(l_m(\mathbf{g}))\| - \chi\right| > \varepsilon\right) + \mathbb{P}\left(\log \|\rho(g_n \dots g_{m+1})\| > 2\chi(n - m)\right)$$

is exponentially small in n by the large deviation estimates for the norm, and the fact that $m \sim (1 - \varepsilon)n$. Thus, to obtain the desired estimate for $\mathbb{P}(\delta(\rho(l_n(\mathbf{g}))) < \varepsilon_n)$ it is enough to estimate the conditional probability

$$\mathbb{P}\left(\delta(\rho(l_n(\mathbf{g}))) < \varepsilon_n \mid \left|\frac{1}{m} \log \|\rho(l_m(\mathbf{g}))\| - \chi\right| \leq \varepsilon, \text{ and } \log \|\rho(g_n \dots g_{m+1})\| \leq 2\chi(n - m)\right).$$

For this, we let $\mathbf{h} = (h_1, \dots, h_m) \in G^m$ be such that

$$(31) \quad \chi - \varepsilon \leq \frac{1}{m} \log \|\rho(l_m(\mathbf{h}))\| \leq \chi + \varepsilon,$$

and we will prove that the conditional probability

$$(32) \quad \mathbb{P}\left(\delta(\rho(l_n(\mathbf{g}))) < \varepsilon_n \mid g_i = h_i \text{ for } i = 1, \dots, m, \text{ and } \log \|\rho(g_n \dots g_{m+1})\| \leq 2\chi(n - m)\right)$$

is bounded by ε_n^K for some K , uniformly in \mathbf{h} satisfying (31). This will give the desired result.

For every \mathbf{h} satisfying (31), there exist two balls $A_m(\mathbf{h})$ and $R_m(\mathbf{h})$ in the Riemann sphere such that $\rho(l_m(\mathbf{h}))(R_m(\mathbf{h})^c) = A_m(\mathbf{h})$ and whose diameter are $\sim \frac{1}{\|l_m(\mathbf{h})\|}$. Indeed, by the KAK decomposition there exist $R, R' \in \text{SU}(2)$ such that $\rho(l_m(\mathbf{h})) = R \begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m^{-1} \end{pmatrix} R'$, where $\sigma_m = \|l_m(\mathbf{h})\|$. Now R and R' act as Euclidean rotations on the Riemann sphere, therefore we can simply put $R_m(\mathbf{h}) = (R')^{-1}(B(0, \sigma_m^{-1}))$ and $A_m(\mathbf{h}) = R(B(\infty, \sigma_m^{-1}))$. In particular, we see that the radii of $A_m(\mathbf{h})$ and $R_m(\mathbf{h})$ are bounded by $e^{(-\chi + \varepsilon)m} = e^{(-\chi + O(\varepsilon))n}$.

Let $\mathbf{g} \in G^{\mathbb{N}}$ be such that $g_i = h_i$ for $i = 1, \dots, m$, and $\|\rho(g_n \dots g_{m+1})\| \leq 2\chi(n - m) \sim 2\chi n \varepsilon$. Then the ball $\rho(g_n \dots g_{m+1})A_m(\mathbf{h})$ has diameter bounded by $e^{(-\chi + O(\varepsilon))n}$. Thus, slightly abusing notation, if we set

$$A_n(\mathbf{g}) := \rho(g_n \dots g_{n-m+1})A_m(\mathbf{h}), \quad R_n(\mathbf{g}) := R_m(\mathbf{h}),$$

then we have that

$$\rho(l_n(\mathbf{g}))(R_n(\mathbf{g})^c) = A_n(\mathbf{g}) \quad \text{and} \quad \text{diam}(A_n(\mathbf{g})), \text{diam}(R_n(\mathbf{g})) \leq e^{(-\chi + O(\varepsilon))n}.$$

We now claim that if the sets $A_n(\mathbf{g})$ and $R_n(\mathbf{g})$ are separated by a distance $\geq \varepsilon_n$, then $\delta(\rho(l_n(\mathbf{g}))) > \varepsilon_n$. Indeed, in this case the two balls are disjoint and the map $\rho(l_n(\mathbf{g}))$ is loxodromic with one fixed point in each ball $A_n(\mathbf{g})$ and $R_n(\mathbf{g})$.

Fix a point $x_0 \in A_m(\mathbf{h})$ and a point $y_0 \in R_m(\mathbf{h})$. If the sets $A_n(\mathbf{g})$ and $R_n(\mathbf{g})$ are not separated by a distance ε_n , then (if n is sufficiently large independently of \mathbf{h}) because their diameter is bounded by $e^{(-\chi + O(\varepsilon))n}$, the point x_0 is mapped under $\rho(g_n \dots g_{m+1})$ into the ball

$B(y_0, 2\varepsilon_n)$. But by the Markovian property, and Lemma A.3 applied to $p = n - m$, we see that this happens only with probability less than $C\varepsilon_p^K$. Since $p = n - m \sim \varepsilon n$, from the choice of possible sequences (ε_n) we get that $\varepsilon_p^K \leq C\varepsilon_n^{K(\varepsilon)}$. Hence for n large enough we have shown that the probability in (32) is bounded by $C\varepsilon_n^{K(\varepsilon)}$, for every \mathbf{h} satisfying (31). The proof is complete. \square

We now study fixed points of pairs of words. We could give more precise estimates in the spirit of Theorem A.1 but those will not be needed.

Theorem A.4. *Let (G, μ, ρ) , (G, μ', ρ) be two admissible families of representations satisfying the exponential moment condition (8).*

Fix $\gamma > 0$. Then for $\mu^{\mathbb{N}} \otimes (\mu')^{\mathbb{N}}$ a.e. $(\mathbf{g}, \mathbf{g}')$, for large enough n , $\rho(l_n(\mathbf{g}))$ and $\rho(l_n(\mathbf{g}'))$ are loxodromic transformations, and the mutual distance between any two of the four associated fixed points is at least $\exp(-\gamma n)$.

Corollary A.5. *Let (G, μ, ρ) , (G, μ', ρ) be two admissible families of representations satisfying the exponential moment condition (8).*

Then for $(\mu^{\mathbb{N}} \otimes (\mu')^{\mathbb{N}})$ -a.e. $(\mathbf{g}, \mathbf{g}') \in (G^{\mathbb{N}})^2$, we have that

$$\frac{1}{2n} \log |\operatorname{tr}[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))] - 2| \xrightarrow{n \rightarrow \infty} \chi(\rho, \mu) + \chi(\rho, \mu').$$

Proof. Fix a small $\gamma > 0$ (in particular small with respect to the Lyapunov exponents) and take $(\mathbf{g}, \mathbf{g}')$ satisfying the conclusion of Theorem A.4, and such that moreover $\frac{1}{n} \log \|l_n(\mathbf{g})\|$ (resp. $\frac{1}{n} \log \|l_n(\mathbf{g}')\|$) is close to $\chi(\rho, \mu)$ (resp. $\chi(\rho, \mu')$). With notation as in the proof of Theorem A.1 we see that

$$[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))](A_n(\mathbf{g}')^c) \subset A_n(\mathbf{g}), \text{ while } \operatorname{dist}(A_n(\mathbf{g}'), A_n(\mathbf{g}')) \gtrsim e^{-n\gamma}$$

(recall that the diameter of these balls is of the order of magnitude of $\exp(-n\chi)$). So we infer that $[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))]$ is a loxodromic element with attracting fixed point in $A_n(\mathbf{g})$ and repelling fixed point in $A_n(\mathbf{g}')$. Inspecting the contraction of a ball of macroscopic size, disjoint from $A_n(\mathbf{g}')$ under $[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))]$ reveals that

$$\|[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))]\| \asymp \|\rho(l_n(\mathbf{g}))\|^2 \|\rho(l_n(\mathbf{g}'))\|^2,$$

and since the fixed points of this commutator are distant from at least $\exp(-\gamma n)$, we conclude from Lemma 2.1 that for large n ,

$$\left| \frac{1}{2n} \log |\operatorname{tr}[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))] - 2| - \frac{1}{2n} \log \|[\rho(l_n(\mathbf{g})), \rho(l_n(\mathbf{g}'))]\| \right| \leq \gamma,$$

which finishes the proof. \square

Proof of the theorem. Observe first that it suffices to prove the theorem for small γ . We will show that the probability that any two of the four sets $A_n(\mathbf{g})$, $R_n(\mathbf{g})$, $A_n(\mathbf{g}')$ and $R_n(\mathbf{g}')$ are closer in distance than $\exp(-\gamma n)$ is exponentially small in n . Then applying the Borel-Cantelli lemma gives the desired result.

We first need to prove that for a.e. $\mathbf{g} \in \mu^{\mathbb{N}}$, the sets $R_n(\mathbf{g})$ tend exponentially fast in distribution to the stationary measure $\check{\nu}$ associated to the inverse random walk. More precisely:

Lemma A.6. *Almost surely the ball $R_n(\mathbf{g})$ converges to a point $R_\infty(\mathbf{g})$ whose distribution is $\check{\nu}$. Moreover, the probability that the distance between R_n and R_∞ is larger than $\exp(-\chi n/4)$ is exponentially small in n .*

Proof. For an element l of $\mathrm{PSL}(2, \mathbb{C})$, and a constant $D > 0$, let us introduce the set

$$R^D(\ell) := \{[x, y] \in \mathbb{P}^1 \mid \|\ell(x, y)\| \leq D \|(x, y)\|\}.$$

Observe that if $1 \leq D \leq \frac{\|\ell\|}{2}$, then $R^D(\ell)$ is contained in a $C^{st} \frac{D}{\|\ell\|}$ -neighborhood of $R^1(\ell)$, and that we can choose $R_n(\mathbf{g})$ to be $R^1(\rho(l_n(\mathbf{g})))$. These considerations are left to the reader.

Because the measure μ has an exponential moment, there is a constant such that

$$\mu(\{\|\rho(g)\| \geq \exp(\chi n/4)\}) \leq C^{st} \exp(-\chi \tau n/4)$$

(τ is the constant appearing in (7)). Therefore, if $E_n \subset G^{\mathbb{N}}$ is defined by

$$E_n = \left\{ \mathbf{g} \in G^{\mathbb{N}}, \forall m \geq n, \|\rho(g_m)\| \leq \exp(\chi m/4) \text{ and } \|\rho(l_m(\mathbf{g}))\| \geq \exp(\chi m/2) \right\},$$

then the measure of E_n is exponentially close to 1 (we use the large deviations estimate for the norm).

Now pick $\mathbf{g} \in E_n$ and let $m \geq n$. Then, on $R_m(\mathbf{g}) = R^1(\rho(l_m(\mathbf{g})))$, we have $\|\rho(l_{m+1}(\mathbf{g}))\| \leq C^{st} \|\rho(g_{m+1})\| \|\rho(l_m(\mathbf{g}))\| \leq \exp(\chi m/4)$ so that $R_{m+1}(\mathbf{g})$ is contained in the $C^{st} \exp(-\chi m/4)$ -neighborhood of $R_m(\mathbf{g})$. We deduce that $R_m(\mathbf{g})$ converges to a point $R_\infty(\mathbf{g})$ for every $\mathbf{g} \in E_n$, and hence a.s. by Borel-Cantelli, and that the distance between $R_n(\mathbf{g})$ and $R_\infty(\mathbf{g})$ is bounded by $A \exp(-\chi n/4)$ for $\mathbf{g} \in E_n$ for some constant A . Of course we can adjust $A = 1$ by introducing appropriate constants in the above reasoning.

To finish the proof, it suffices to verify that the distribution of R_∞ is given by the stationary measure $\check{\nu}$. For this, we argue as in Step 2 of the proof of Theorem 3.9, by observing that the point $R_\infty = \lim R(l_n(\mathbf{g}))$ only depends on the tail of the sequence $l = (l_n(\mathbf{g}))$ – or equivalently of $r = (r_n(\check{\mathbf{g}}))$, with $\check{\mathbf{g}} = (g_n^{-1})$ – and satisfies the equivariance property (20), with respect to the inverse process. Thus, R_∞ defines a map from the Poisson boundary $P(G, \check{\mu})$ to \mathbb{P}^1 which is ρ -equivariant. The distribution of the point R_∞ is hence given by the stationary measure $\check{\nu}$. \square

With this at hand, let us conclude the proof of the theorem. Fix $\gamma < \chi/2$. Recall from Theorem A.1, that with probability exponentially close to 1, $\|l_n(\mathbf{g})\| \geq \exp((\chi - \varepsilon)n)$ and $A_n(\mathbf{g})$ and $R_n(\mathbf{g})$ are $\exp(-\gamma n)$ -separated. Fix a pair of such elements \mathbf{g}, \mathbf{g}' , and let us show that the probability that the balls $A_n(\mathbf{g})$ and $R_n(\mathbf{g})$ are not $\exp(-\gamma n)$ -separated from both $A_n(\mathbf{g}')$ and $R_n(\mathbf{g}')$ is exponentially small.

Let us first estimate the probability that $R_n(\mathbf{g})$ intersects the $\exp(-\gamma n)$ -neighborhood of $A_n(\mathbf{g}') \cup R_n(\mathbf{g}')$. This implies that the point $R_\infty(\mathbf{g})$ is $C^{st} \exp(-\gamma n)$ -close to $A_n(\mathbf{g}') \cup R_n(\mathbf{g}')$, hence belongs to a union of two fixed balls of radii $C^{st} \exp(-\gamma n)$ (recall that $\gamma < \chi/2$). By Lemma A.6 and the Hölder regularity property of the stationary measure, we conclude that this event happens with probability bounded by $C^{st} \exp(-\gamma \eta n)$ for some $\eta > 0$.

To bound the probability that $A_n(\mathbf{g})$ intersects the $\exp(-\gamma n)$ -neighborhood of $A_n(\mathbf{g}') \cup R_n(\mathbf{g}')$, we simply reverse the random walk on G , and use the fact that the distribution of $A_n(\check{\mathbf{g}})$ is the same as that of $R_n(\mathbf{g})$. \square

APPENDIX B. A NUMBER-THEORETIC ESTIMATE

Our purpose here is to prove the following result. We thank P. Philippon for explaining it to us³.

³We were informed by S. Boucksom that in case V is defined over \mathbb{Q} , [BC, Lemma 2.6] gives the same result with the term $\deg(P) \log \deg(P)$ replaced by $\deg(P)$.

Proposition B.1. *Let V be an irreducible affine algebraic variety in \mathbb{C}^n , defined over $\overline{\mathbb{Q}}$, and $U \Subset V$ be a relatively compact open subset. Then there exists a constant $\delta > 0$ (depending on V, U, n) such that if $P \in \mathbb{Z}[X_1, \dots, X_n]$ is any polynomial with integer coefficients, then*

- either $P|_V = 0$
- or $\sup_U |P| \geq \delta^{\deg(P) \log \deg(P) + \log H(P)}$, where $H(P)$ is the maximum modulus of the coefficients of P .

Here is the precise corollary that we need.

Corollary B.2. *Let V, U, P be as in Proposition B.1. Then there exists a constant $\delta > 0$ such that the following alternative holds*

- either $P|_V$ is constant
- or $\text{var}_U P \geq \delta^{\deg(P) \log \deg(P) + \log H(P)}$, where $\text{var}_U P = \sup_{x, y \in U} |P(x) - P(y)|$.

To obtain the corollary, it is enough to apply the proposition to $\tilde{P} : (x, y) \mapsto P(x) - P(y)$ restricted to $\tilde{V} := V \times V \subset \mathbb{C}_x^n \times \mathbb{C}_y^n$.

For the matter of proving the proposition we briefly introduce a few concepts from number theory; the reader is referred to [W] for details. If $P \in \mathbb{Z}[X_1, \dots, X_n]$, the *usual height* $H(P)$ of P is the maximum modulus of its coefficients. Let now $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number, and $P \in \mathbb{Z}[X]$ be its minimal polynomial. By definition, the degree $\deg(\alpha)$ equals $\deg(P)$ and we let $H(\alpha) := H(P)$. We do not need to define precisely the *height* $h(\alpha)$ of α , but only note that it satisfies $\left| h(\alpha) - \frac{1}{\deg(\alpha)} \log H(\alpha) \right| \leq C$, where C is a universal constant ≤ 2 . If $\alpha = p/q$ is rational, $h(\alpha) = \log \max(|p|, |q|)$. Also $h(\cdot)$ behaves well under the operation of taking sums and products of algebraic numbers.

Another useful property is that if P_1, P_2 are polynomials in $\mathbb{Z}[X]$ with respective degree d_1, d_2 , then $H(P_1 P_2) \geq 2^{-d_1 d_2} (d_1 d_2 + 1)^{-1/2} H(P_1) H(P_2)$. From this we infer that if $\alpha \in \overline{\mathbb{Q}}$, and $P \in \mathbb{Z}[X]$ is any polynomial such that $P(\alpha) = 0$, then there exists a constant $C = C(\deg(P))$ depending only on $\deg(P)$ such that $h(\alpha) \leq \log H(P) + C$. Furthermore, if now α satisfies an algebraic equation of the form $P(\alpha) = 0$, where $P \in \mathbb{Q}(\beta)[X]$, with $\beta \in \overline{\mathbb{Q}}$, then $h(\alpha) \leq C(\deg(P), \beta)(h(\beta) + 1)$, where $h(\beta)$ denotes the maximum height of the coefficients of P . To see this, just observe that the product of the Galois conjugates of P is an annihilator of α belonging to $\mathbb{Q}[X]$ and estimate its degree and coefficients.

When V is merely a point, the estimate in Proposition B.1 is classical and known as the *Liouville inequality* (see [W, Proposition 3.14]). We need to state it precisely: if x_1, \dots, x_n are algebraic numbers, and $P \in \mathbb{Z}[X_1, \dots, X_n]$ is a polynomial not vanishing at $x = (x_1, \dots, x_n)$, then there exists a constant c depending only on the dimension n such that

$$(33) \quad |P(x_1, \dots, x_n)| \geq e^{-cD(\deg(P) \max_i h(x_i) + \log H(P))}, \text{ where } D = [\mathbb{Q}(x_1, \dots, x_n) : \mathbb{Q}].$$

Proof of the proposition. Throughout the proof the notation $a \lesssim b$ means $a \leq Cb$ where C is a constant independent of P (and similarly for $a \gtrsim b$). The main step is to prove that if $P|_V \neq 0$, then there exists an algebraic point $x = (x_1, \dots, x_n) \in V$, such that P does not vanish at x and furthermore $\max_i \deg(x_i) \lesssim 1$ and $\max_i h(x_i) \lesssim \log \deg(P)$. Then, applying the Liouville inequality (33) to $|P(x)|$ gives the result.

To show the existence of such a point x , we use a projection argument. We fix a linear projection $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^{\dim V}$, defined over \mathbb{Q} , in general position with respect to V . Let

$\Sigma = \{P = 0\} \cap V$, which is a proper subvariety of V . The projection $\pi(\Sigma) \subset \mathbb{C}^{\dim(V)}$ is a hypersurface of degree at most $\deg(V) \deg(P)$.

Let $k = \dim(V)$ and let $B \subset \mathbb{C}^k$ be a ball contained in $\pi(U)$. We claim that there exists a rational point $y \in \mathbb{Q}^k$ such that $y \in B \setminus \pi(\Sigma)$ and $h(y) \lesssim \log \deg(P)$ (abusing slightly, here we put $h(y) = \max_i h(y_i)$, where the y_i are the coordinates of y). If $k = 1$ this is obvious since $\pi(\Sigma)$ contains at most $\deg(V) \deg(P)$ points while $\#\{y \in B \cap \mathbb{Q}, h(y) \leq h\} \gtrsim e^h$. Now if $k = 2$, $\pi(\Sigma)$ contains at most $\deg(V) \deg(P)$ lines, so there is a line over \mathbb{Q} defined by an equation of height $\lesssim \log(\deg(V) \deg(P))$ not contained in $\pi(\Sigma)$, and in this line we are back to the previous case $k = 1$. The general case follows by induction.

Finally, to obtain the desired x we simply lift y to V . Therefore, x is an intersection point between V and the fiber $\pi^{-1}(y)$ of the projection π through y , which is a $(n - k)$ -plane parallel to some fixed rational direction and passing through y . To estimate the degree and height of x , we work in a projective space \mathbb{P}^n compactifying \mathbb{C}^n . Since π is a linear projection defined over \mathbb{Q} , Bézout's theorem implies that $\deg(x) \leq \deg(V) \deg(\pi^{-1}(y)) = \deg(V) \lesssim 1$. Similarly, there are Bézout-type theorems for the height of intersections of projective varieties (see [BGS] or [Ph, Thm 3]). In our case, it expresses as

$$h(V \cdot \pi^{-1}(y)) \leq h(V) \deg(\pi^{-1}(y)) + h(\pi^{-1}(y)) \deg(V) + c \deg(V) \deg(\pi^{-1}(y)),$$

where c is a dimensional constant. Here $V \cdot \pi^{-1}(y)$ is a 0-dimensional cycle containing x so that $h(x) \leq h(V \cdot \pi^{-1}(y))$. The precise definition of the height of an algebraic subvariety is delicate, and differs slightly among authors. Fortunately, these definitions differ from at most an additive constant depending on the dimension. To fix the ideas let us say that we define h according to [BGS]. Since V is a fixed variety, to obtain the desired estimate on $h(x)$ we just need to check that $h(\pi^{-1}(y)) \lesssim \log \deg(P)$.

To see this, we simply note that the height of a projective subspace is the height of its image under the Plücker embedding of the corresponding Grassmanian (see the remarks about Proposition 4.1.2. in [BGS]). Here $\pi^{-1}(y)$, viewed as a projective subspace in \mathbb{P}^n , lifts to a linear subspace of dimension $n - k + 1$ in \mathbb{C}^{n+1} . We can choose a basis v_1, \dots, v_{n-k+1} made of vectors of height $\lesssim h(y)$, and the Plücker image of $\pi^{-1}(y)$ is $v_1 \wedge \dots \wedge v_{n-k+1}$. Since the height is subadditive under multiplication, we see that the coordinates of this vector are $\lesssim h(y) \lesssim \log \deg(P)$, therefore $h(\pi^{-1}(y)) \lesssim \log \deg(P)$, which finishes the proof. \square

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